

Superalgebraisation of BFV-BRST quantisation

(Stuart Paul)

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Abstract

In recent years it has been shown that, for the BFV-BRST extended phase-space quantisation of relativistic systems, hidden higher symmetry superalgebras are relevant. We examine the relationship between the BFV-BRST extended phase space construction for several cases and the structure of certain classes of representations of orthosymplectic superalgebras $osp(D, 2/2)$ and their inhomogeneous extensions. Our motivation is to show, by way of several examples, how the hidden symmetry is manifested in the BFV-BRST method.

In this class of superalgebras the physical particle states (that is, irreducible representations of the Poincaré algebra, carried on appropriate types of relativistic wavefunctions) appear as the resolution of the BRST complex. This complex is naturally provided via an appropriate nilpotent odd generator (the BRST operator).

Specific examples studied are the relativistic scalar and spinning particle, as well as a generalisation of the scalar particle in two dimensions, which we show to be associated with the exceptional superalgebra $D(2, 1; \alpha)$. This superalgebra is isomorphic to $osp(2, 2/2)$ for $\alpha = 1$.

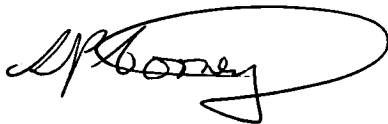
Declaration

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A handwritten signature in black ink, appearing to read 'Stuart Corney', enclosed within a large, loopy oval stroke.

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'All men dream, but not equally. Those who dream by night in the dusty recesses of their minds wake in the day to find that it was vanity; but the dreamers of the day are dangerous men, for they may act their dreams with open eyes, to make it possible'

T. E. Lawrence

'The amount of theoretical work one has to cover before being able to solve problems of real practical value is rather large, and is likely to become more pronounced in the theoretical physics of the future'

P. A. M. Dirac, 1930

'I am enough of an artist to draw freely upon my imagination. Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world'

Albert Einstein

'That first egg was named Thought ... Elemental forces caused the egg to hatch; from it then came a stone monkey. The nature of Monkey was irrepressible!'

Wu Cheng-en

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Chapter 1

Preface

Quantum mechanics is difficult to describe and understand; almost any undergraduate student will assure you of this. Mathematically it is also difficult in many situations to formulate a fully covariant quantum theory, and in fact only fairly simple quantum field theories have been successfully constructed. On the other hand, classical mechanics, which is the limiting case of quantum theory when \hbar tends to zero, is much easier to visualise and much simpler to calculate. But of what interest is classical mechanics when we know that nature is inherently quantum mechanical? This is where quantisation plays an important role; quantisation is, quite simply, the process of transforming from an appropriately formulated classical theory to a quantum theory. This thesis explores the algebraic aspects of the quantisation of constrained systems.

The process of the quantisation of systems with constraints has had a long development, with many important steps taken since the seminal monographs of Dirac[1]. Techniques have been introduced to handle gauge theories, such as non-abelian Yang-Mills-Shaw theory and (linearised) gravity, which have culminated in the demonstration of global supersymmetries[2, 3] for such systems, under which gauge and ghost degrees of freedom transform. These degrees of freedom even play a role at the level of classical dynamics with finitely many degrees of freedom. Further, it has been shown that in certain cases it is possible to further unify these ‘quantisation’ supersymmetries with other symmetries possessed by

the system, particularly those associated with the constraint algebra. In these cases the entire state space may be constructed from the representation theory of the enlarged algebra. The ultimate goal of work of this nature is to develop sufficient understanding of the gauge symmetries themselves, the nature of their graded extensions, and the associated representation theory, so that admissible quantisation(s) may be implemented systematically (and covariantly) at this algebraic level.

In this thesis some steps are taken in this direction; the attitude we have adopted is that the general principles of the algebraic version of the quantisation program emerge from detailed consideration of particular case studies. To this end we present three examples of different particles, as well as ideas for a possible extension of the method into two-time physics [4].

The structure of this thesis is as follows;

In chapter 2 we present the necessary background material and techniques that will be needed in later chapters. We introduce pseudo-classical Hamiltonian and Lagrangian mechanics, and then define a constrained or singular system. Primary and secondary, as well as first and second class constraints are discussed. This leads us to the problem of second class constraints and so to the introduction of Dirac brackets. We then outline the method of Dirac quantisation, upon which nearly all forms of quantisation are based. Finally we look at Grassmann coordinates, which lead to the idea of superfunctions and Poisson brackets.

In chapter 3 we summarise the general techniques of BFV-BRST quantisation; that is we define an extended phase space and define the BRST [2, 3] operator. Next we outline the quantisation method developed by Batalin, Fradkin and Vilkovisky [5, 6, 7], which has become known as BFV-BRST quantisation. In the second half of chapter 3 we introduce the class of $osp(d, 2/2)$ algebras, and show how they relate to BFV-BRST quantisation.

In chapter 4 we present our first example, that of the scalar particle. The material presented in this chapter was previously published [8] using a method of produced representations, but here we advance upon this work by examining the

BFV-BRST quantisation of the scalar particle model via a scalar representation of $iosp(d, 2/2)$. We also give a background explanation of the scalar particle which includes a description of its dynamics and presents it in various forms.

Continuing on from chapter 4, in chapter 5 we present the covariant BFV-BRST quantisation of a spinning particle, one of the simplest examples of a supersymmetric system. This chapter is more extensive than that for the scalar particle, due to the presence of anti-commuting coordinates in the classical action. The presence of these anti-commuting (Fermionic) coordinates requires the introduction of a graded Clifford algebra, which will allow us to split the Dirac wavefunctions and thus apply the realisation on $2^{d/2}$ Dirac spinors. A brief explanation of the properties of the spinning particle is also given, by way of introduction.

A further step is taken in chapter 6; here we take an algebraic approach and extend the quantisation superalgebra $iosp(d, 2/2)$ into a more general classical simple Lie superalgebra, namely $D(2, 1; \alpha)$. The idea behind this is to develop a characterisation of admissible spacetime BFV-BRST extended supersymmetries in various dimensions (using $d = 2$ as an example). In this chapter we review the properties of the exceptional superalgebra $D(2, 1; \alpha)$ before proceeding to construct a BFV-BRST quantised particle which exists in $D(2, 1; \alpha)$. Finally we shall take a backward step and derive the classical action for such a particle.

In chapter 7 we review two-time physics as formulated by Bars [4], and construct a BRST equivalent form of a two-time particle. Our motivation for doing this is to confirm the results of Bars *et al.* from a BRST perspective. By necessity we also examine methods of removing second class constraints from classical systems. This chapter contains unfinished work and as such has several suggestions for future research into this branch of mathematical physics.

Finally, chapter 8 contains the conclusions reached in this project, as well as ideas for future work and improvements upon that done so far.

Chapter 2

Introductory Material

This chapter provides the background knowledge and techniques necessary in order to understand the work in this thesis. In this respect it is a review of the literature, however it is by no means an exhaustive survey. We seek to present the necessary ideas in a comprehensible manner, with references to further reading and proofs for those who wish to follow things up.

The outline of this chapter is as follows. We shall begin by describing the correct method of expressing a classical system so as to allow quantisation, and some of the difficulties that may arise in so doing. Secondly we shall examine the quantisation method proposed by Dirac [1]. This forms the basis of many of the later methods, including that of Batalin, Fradkin and Vilkovisky (BFV) [5, 6, 7]. We shall then consider Grassmann algebras and Grassmann coordinates, the properties they yield and how they relate to quantum mechanics. In chapter 3 we will apply the knowledge gained in this chapter to the process of BFV-BRST quantisation. Most of this work is standard and can be found in many textbooks; the first two sections are covered in [1, 9, 10, 11, 12], a comprehensive treatment of section 2.3 can be found in [13, 9, 14] whilst 3.1 can be found in [9, 14].

2.1 The Hamiltonian Method

If we can put a classical theory into an appropriate Hamiltonian form, then we can always apply certain standard rules so as to arrive at a description of the corresponding quantum theory. This statement is the fundamental tenet of Hamiltonian quantisation and motivates the work carried out in this section. The starting point for Hamiltonian quantisation is an action principle.

$$S = \int \mathcal{L} dt, \quad (2.1)$$

where \mathcal{L} is the classical Lagrangian of the system. We must then transform this Lagrangian into a Hamiltonian. The reason for starting with a Lagrangian rather than a Hamiltonian has to do with special relativity; it is not easy to formulate conditions for a theory to be relativistic purely in terms of a Hamiltonian; on the other hand, if we ensure that the action integral is invariant, then we guarantee a relativistic solution. We can easily construct an invariant action integral, which thus leads to equations of motion complying with special relativity. Any developments from this action integral (*e.g.* the Hamiltonian we derive) will therefore also comply with the special theory of relativity.

As this thesis deals with quantisation of particles, and not fields, we consider a finite number of degrees of freedom, whose generalised coordinates we denote $q_n(t)$, $n = 1, \dots, N$ and their corresponding velocities $\dot{q}_n(t) = \frac{\partial q_n}{\partial t}$. It is a straight forward matter to generalise to an infinite number of degrees of freedom, and thus describe fields. The Lagrangian is a function $\mathcal{L} = \mathcal{L}(q_n, \dot{q}_n)$ of the coordinates and velocities. By calculating the variation of \mathcal{L} with q , and applying the action principle to the action S , we can derive[1] the Euler-Lagrange equations of motion

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_n} \right) - \frac{\partial \mathcal{L}}{\partial q_n} = 0. \quad (2.2)$$

To express the system in Hamiltonian formalism we need to introduce the momenta p_n defined by

$$p_n = \frac{\partial \mathcal{L}}{\partial \dot{q}_n}. \quad (2.3)$$

In the usual dynamical theory (*i.e.* regular systems) the momenta are assumed to be independent functions of the velocities and thus the Hessian of the Lagrange function has a non vanishing determinant

$$\det \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_{n_1} \partial \dot{q}_{n_2}} \neq 0.$$

When this determinant vanishes identically, the system is said to be singular, or constrained. It is the constrained case which is the subject of this thesis, and so we shall assume that our system is singular. The condition that the determinant of the Hessian vanishes is equivalent to saying that the matrix defined by $\partial p_n / \partial \dot{q}_m$ has zero modes, *i.e.* velocities \dot{q}_n are not all uniquely determined in terms of q_n and p_n only. In other words, in the (q_n, \dot{q}_n) phase space not all momenta are locally independent and so there must exist certain relations

$$\phi_m(q, p) = 0, \quad m = 1, \dots, M \leq N, \quad (2.4)$$

that follow from the definition (2.3) of the momenta. If the Hessian matrix has rank R then the system is said to be irreducible, if $R = N - M$. This corresponds to all the constraints being independent. If not all the constraints are independent then the system is said to be reducible. In this thesis we shall only consider irreducible systems, although the theory can be developed for reducible systems in a similar fashion. When the p_n are replaced by their definition (2.3), equation (2.4) reduces to an identity. The conditions (2.4) are called primary constraints, so as to emphasize that they are not derived from the equations of motion, and that they imply no restriction on the coordinates and their velocities. The exact form of the primary constraints is dependent upon the particular Lagrangian used to represent the system, and thus different Lagrangians, which yield identical equations of motion, can also lead to different primary constraints.

We are now in a position to define the Hamiltonian as follows

$$H = \sum_n p_n \dot{q}_n - \mathcal{L}. \quad (2.5)$$

By considering variations with respect to q_n and \dot{q}_n , it can be shown that H is

independent of $\delta\dot{q}_n$, in fact

$$\delta H = \sum_n \left[\delta p_n \dot{q}_n - \frac{\partial \mathcal{L}}{\partial q_n} \delta q_n \right].$$

This independence from $\delta\dot{q}$ allows us to reformulate the Hamiltonian purely in terms of q_n and p_n , independent of the velocities \dot{q}_n . A crucial feature of the Hamiltonian as we have defined it, is that it is *not unique*; we may add to it any linear combination of the primary constraints and still arrive at the same equations of motion once the constraints are imposed

$$H^* = H + \sum_m u_m(q_n, p_n) \phi_m(q_n, p_n), \quad (2.6)$$

where the u_m are the Lagrange multipliers for the constraints. The Lagrange multipliers are arbitrary functions of the phase space variables q_n, p_n , and are such that the summation in the above equation is always a real quantity.

By applying the standard methods of the calculus of variations to the Hamiltonian with constraints H^* , we can derive the Hamiltonian equations of motion

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial H}{\partial q_n} - u_m \frac{\partial \phi_m}{\partial q_n}. \quad (2.7)$$

Note that the first of these equations gives a method of returning to the position velocity description of a system.

The equations of motion can again be rewritten by introducing Poisson Brackets; given two functions on phase space f and g , the Poisson Bracket between the two is given by

$$\{f, g\} = \frac{\partial f}{\partial q_n} \frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n} \frac{\partial g}{\partial q_n}. \quad (2.8)$$

By definition, the Poisson bracket is anti-symmetric, linear (when multiplied by a constant), distributive, associative and obeys the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

We can easily show that the only non-zero Poisson brackets between q_n and p_n are $\{q_n, p_m\} = \delta_{nm}$.

The equations of motion can now be written in Poisson bracket form as

$$\dot{q}_n = \{q_n, H^*\}, \quad \dot{p}_n = \{p_n, H^*\}.$$

In fact we can generalise this to write the time evolution for any function of phase space f

$$\begin{aligned} \dot{f} &= \{f, H + \sum_m u_m \phi_m\}, \\ &= \{f, H\} + \sum_m (\{f, u_m\} \phi_m + u_m \{f, \phi_m\}), \end{aligned}$$

The second term in this expansion disappears as $\phi_m = 0$ and so we can write the time evolution of any function as

$$\dot{f} = \{f, H\} + u_m \{f, \phi_m\}. \quad (2.9)$$

A key point of which one must be mindful when using the Poisson Bracket formalism is that we must not impose the constraints until after we have calculated the Poisson brackets. To make clear this distinction we define [1] the concept of weak equality (\approx) and write $\phi_m \approx 0$, with the understanding that one can only enforce weakly equal equations after all relevant Poisson brackets have been calculated. An alternative way of viewing this is that weakly equal equations correspond to an expression that is numerically restricted to zero, but does not vanish identically throughout phase space. This means, in particular, that it has non-zero Poisson brackets with the canonical variables.

More generally, two functions f, g , that coincide once the constraints are imposed, are said to be weakly equal, and we write $f \approx g$. An equation that holds throughout all phase space, independent of the constraints, is said to be strongly equal ($=$). Thus we have

$$f \approx g \Leftrightarrow f - g = \sum_m c_m \phi_m,$$

for some arbitrary functions c_m .

Using the concept of weak equality we can write the equation of motion of an arbitrary function $\dot{f} \approx \{f, H^*\}$. A basic consistency requirement of the system

is that the primary constraints ϕ_m be conserved in time, thus by setting $f = \phi_m$ and knowing that $\dot{\phi}_m = 0$ we arrive at the basic consistency condition for the system

$$0 = \dot{\phi}_m \approx \{\phi_m, H^*\} = \{\phi_m, H\} + \sum_{m'} u_{m'} \{\phi_m, \phi_{m'}\}. \quad (2.10)$$

This equation must be satisfied (assuming the Lagrange function always yields consistent solutions), and so we are faced with three possibilities. Firstly, (2.10) may be identically satisfied (once the primary constraints are imposed), *i.e.* it leads to the situation $0 = 0$. Another possibility is for (2.10) to lead to an equation independent of the functions u_m , but giving a relationship between the q_n and p_n . Such an equation must be independent of the primary constraints ϕ_m , and so can be written

$$\chi_m(q, p) = 0.$$

Finally we may get the situation where (2.10) leads to an equation involving conditions on the u_m .

If the second situation arises, then we effectively have a new set of constraints χ_m , which we call secondary constraints. Thus the primary constraints are those that are consequences only of the equations that define the momentum, whilst secondary constraints are those that arise due to the equations of motion (and thus the Lagrangian) as well. Each secondary constraint yields a new consistency equation, as we still require $\dot{\chi}_m = 0$, *i.e.*

$$0 = \dot{\chi}_m \approx \{\chi_m, H\} + \sum_{m'} u_{m'} \{\chi_m, \phi_{m'}\}.$$

Once again this equation must fall into one of the three categories listed above and so may yield another set of secondary constraints, which may in turn yield yet another set of constraints, and so on. This process must be followed until all secondary constraints have been identified.

The primary and secondary characterisation of constraints depends on which action is used to describe a given system: different Lagrangians which lead to the same equations of motion will lead to identical constraints, however their characterisation as primary or secondary may change. As such the distinction

between primary and secondary constraints is of little importance, and so we shall now write all constraints in the form

$$\phi_j(q_n, p_n) \approx 0, \quad j = 1, \dots, K, K+1, \dots, K+L = J, \quad (2.11)$$

where we have J constraints, made up of K primary and L secondary.

Let us now consider the third type of consistency equation, that is those that place explicit conditions on the u_m

$$\{\phi_j, H\} + \sum_m u_m \{\phi_j, \phi_m\} = 0, \quad j = 1, \dots, J. \quad (2.12)$$

This equations leads to a set of linear equations of u_m in terms of q_n and p_n . The general solution to such a system is

$$u_m = U_m + \sum_a v_a V_{am},$$

where U_m is a specific solution to (2.12), V_{am} , $a = 1, \dots, A$, is the set of all solutions to the corresponding homogeneous equation $\sum_m V_m \{\phi_j, \phi_m\} = 0$, and $v_a(t)$ is a completely arbitrary function of time. The total Hamiltonian can now be written

$$\begin{aligned} H_T &= H_0 + \sum_m \left(U_m \phi_m + \sum_a v_a V_{am} \phi_m \right), \\ &= H + \sum_a v_a \phi_a, \end{aligned} \quad (2.13)$$

where

$$\phi_a = \sum_m V_{am} \phi_m. \quad (2.14)$$

In terms of this total Hamiltonian, we still have the original equations of motion.

Finally, some of the functions u_m are uniquely determined using (2.12) (namely those for which the only solutions correspond to $V_{am} = 0$ for all values of j). However for the remaining functions (when $V_{am} \neq 0$), only part of each u_m is determined by (2.12), as there is also the v_a which are arbitrary functions of time. Therefore the general solution to the Hamiltonian equations of motion obtained using H_T depend on arbitrary functions of time. This arbitrariness is a feature of

singular Hamiltonian systems and is due to the fact that their description includes dependent degrees of freedom.

As we have said, whether a constraint is primary or secondary depends on the choice of the Lagrangian for a system. There is however a characterisation of constraints which is independent of the Lagrange function. Define any dynamical variable V , a function of q and p , as first class if

$$\{V, \phi_j\} \approx 0 = \sum_k C_j^k \phi_k, \quad (2.15)$$

for some phase space function C_j^k . Otherwise V is said to be second class. Thus V is second class if the Poisson bracket of at least one of the constraints does not vanish weakly.

An important feature of the first class property is that it is closed, *i.e.* the Poisson bracket of two first class functions is also first class. Note that the Hamiltonian (2.13) and the constraints ϕ_a are first class, as can be seen through (2.11) and (2.14) respectively. Thus the total Hamiltonian can be expressed as the sum of a first class Hamiltonian H^* and the first class constraints ϕ_a multiplied by arbitrary coefficients.

2.1.1 Dirac Brackets

We shall now turn our attention to second class constraints. An example of a pair of second class constraints is $p_1 = 0, q_1 = f(q_r, p_r)$, $r = 2, \dots, N$, *i.e.* the first position variable is a function of the remaining position and momentum variables. As can be seen by this example, a second class constraint corresponds to a degree of freedom which is not physically relevant (redundant for a description of the system). In order to quantise a system with second class constraints it is necessary to pick out which degrees of freedom are unimportant and formulate the Poisson brackets such that they depend only on the remaining coordinates. In terms of these new Poisson brackets and coordinates we can then carry out Dirac quantisation.

Dirac [1] set out a general method for carrying out the reduction of a system

to one with first class constraints, which we shall now outline. Suppose we have constraints $\phi_j \approx 0$, some of which are first class, some second class. Any linear combination of constraints

$$\phi'_j = \sum_{j'} A_{jj'} \phi_{j'},$$

where A is invertible, must also be a constraint. Thus we can replace our original constraints by linear combinations of the constraints which are chosen in such a way as to maximise the number of first class constraints. The remaining second class constraints (which cannot be expressed as linear combinations of any of the first class constraints) we denote

$$\chi_s \approx 0.$$

We form the matrix of Poisson brackets of second class constraints

$$\Delta_{ss'} = \{\chi_s, \chi_{s'}\}.$$

This matrix is non-singular, as singularity would imply that some of these constraints χ_s can be expressed as a linear combination of other constraints $\chi_{s'}$, but we have excluded this possibility by our construction of the second class constraints. As $\Delta_{ss'}$ is non-singular its inverse exists, and may be denoted

$$C^{ss'} = (\Delta)^{ss'},$$

with

$$\sum_{s'} C^{ss'} \{\chi_{s'}, \chi_{s''}\} = \delta_{s''}^s = \sum_{s'} \{\chi_{s''}, \chi_{s'}\} C^{s's}.$$

We can now define the Dirac bracket of two phase space quantities f and g as

$$\{f, g\}_D = \{f, g\} - \sum_{ss'} \{f, \chi_s\} C^{ss'} \{\chi_{s'}, g\}. \quad (2.16)$$

It is straightforward to show that Dirac brackets are generalised Poisson brackets, in that they satisfy the same defining rules of anti-symmetry, associativity and distributivity, as well as the Jacobi identity. Obviously, if a system contains only first class constraints then the Dirac and Poisson brackets will be identical, but in general the exact values of the two different functions may vary.

The equation of motion of an arbitrary phase space quantity f using Dirac brackets can be calculated as

$$\begin{aligned}\{f, H\}_D &= \{f, H\} - \sum_{ss'} \{f, \chi_s\} C^{ss'} \{\chi_{s'}, H\}, \\ &\approx \{f, H\},\end{aligned}\tag{2.17}$$

as the Poisson bracket of any second class constraint and the first class Hamiltonian vanishes weakly. Likewise, the Dirac bracket of an arbitrary phase space quantity f with any of the second class constraints χ_s is

$$\begin{aligned}\{f, \chi_s\}_D &= \{f, \chi_s\} - \sum_{s's''} \{f, \chi_{s'}\} C^{s's''} \{\chi_{s''}, \chi_s\}, \\ &= \{f, \chi_s\} - \sum_{s'} \{f, \chi'_{s'}\} \delta_{s's} = 0.\end{aligned}$$

Thus the Dirac bracket achieves precisely the result that we are after; redefining the Poisson brackets in such a way that the redundant degrees of freedom related to a second class constraint are not included. The second class constraints may now be imposed exactly, and not as weak equalities, even before calculating the Dirac brackets. Therefore, at least in principle,* it is always possible in the Hamiltonian formalism to reduce any constrained singular system to one with first class constraints only.

According to Darboux's theorem, any system, with a consistent symplectic structure, containing only first class constraints (or one in which the second class constraints have been removed by introduction of Dirac brackets) can be described in terms of a single set of phase space variables z_A , which parametrise the subspace of the original phase space (q_n, p_n) defined by the second class constraints $\chi_s = 0$. In a system where there are no second class constraints (and thus Dirac brackets are not necessary) the z_A , $A = 1, \dots, 2N$ can simply be written

$$z_A = q_1, \dots, q_N, p_1, \dots, p_N.$$

*We say 'in principle' as there is no guarantee that an explicit representation of the Dirac brackets exists [9]

The geometrical structure of phase space is determined by the Dirac brackets

$$\{z_A, z_B\}_D = C_{AB},$$

where the C_{AB} are functions on phase space. Time evolution is determined by the Dirac brackets of any phase space quantity with the first class Hamiltonian H . Finally the system is subject to first class constraints, assumed to be regular, and henceforth denoted ϕ_a , where

$$\{\phi_a, \phi_b\}_D = C_{ab}^c \phi_c, \quad (2.18)$$

$$\{H, \phi_a\}_D = V_a^b \phi_b. \quad (2.19)$$

Here for the first time we have used the summation convention for repeated indices (one raised and one lowered). We shall continue to use this convention throughout this thesis.

An example of the use of Dirac brackets can be seen in section 7.3. In this section we shall also discuss other methods that can be employed to convert a system with second class constraints into a first class system.

2.1.2 First class constraints and gauge transformations

This section is a summary of what appears in [9] on this subject. We recommend reading this reference if further details are required. The presence of arbitrary functions v_a in the total Hamiltonian (2.13) implies that not all the z_A are physically observable, *i.e.* there is more than one set of values of the canonical variables that represent a given physical state. To see how this comes about, consider the following argument: given an initial set of z_A at time t_1 , we would expect the equations of motion to fully determine the physical state at time $t_2 \neq t_1$. Thus any ambiguity in the definition of the canonical variables at t_2 should be physically irrelevant.

The coefficients v_a are arbitrary functions of time, and so the phase space variables $z_A(t_2)$ will depend on the choice of v_a in the interval $[t_1, t_2]$. If we consider $t_2 = t_1 + \delta t$, and let v_a, v'_a be two different choices of the arbitrary

functions, then at t_2 a dynamical variable F which is defined at t_1 can differ by

$$\delta F = (v_a - v'_a)\delta t\{F, \phi_a\}. \quad (2.20)$$

But we have said that there should be no ambiguity in the evolution of a variable, and therefore the physical state at time t_2 must be independent of the above transformation. Borrowing terminology from the theory of gauge fields, we say that the first class constraints generate gauge transformations of the system. The gauge transformations are independent if and only if the constraints ϕ_a are irreducible.

The following two results involving gauge transformations also hold

- The Poisson bracket $\{\phi_a, \phi_a\}$ of any two first class constraints generate a gauge transformation.
- The Poisson bracket $\{\phi_a, H_T\}$ of any first class constraint with the total first-class Hamiltonian generates a gauge transformation.

Finally, the constructions used to prove that ϕ_a and $[\phi_a, H_T]$ generate gauge transformations rely on the assumption that the time t is a physical observable. In reparametrisation invariant systems one may take the alternative view that the gauge arbitrariness indicates that the time itself is not observable. In this case one of the arbitrary functions v_a is then associated with a reparametrisation $t \rightarrow \tau(t)$ of the time variable. This scenario will be seen in the later chapters. The same results arise from both interpretations.

2.2 Dirac quantisation

Having now obtained a fundamental Hamiltonian description of a relativistic singular system which contains only first class constraints, it is straightforward to carry out the quantisation procedure of Dirac. Note that from now on we shall use standard ‘curly brackets’ without the subscript D in our calculations, with the understanding that these are in fact Dirac brackets if the system originally contained second class constraints.

Associated with the phase space degrees of freedom z_A we have operators z_A acting on a Hilbert space. Being operators, they do not generally commute, and in fact the commutation relations of the system are obtained by replacing the Poisson brackets of the classical degrees of freedom with commutation relations of the corresponding operators, divided by $i\hbar$. Thus

$$\{z_A, z_B\} = C_{AB} \rightarrow [z_A, z_B] = z_A z_B - z_B z_A = i\hbar C_{AB}. \quad (2.21)$$

We must also introduce an additional structure on the Hilbert space, namely that of an inner product $\langle \cdot | \cdot \rangle$. This inner product must satisfy two properties: an operator corresponding to a real classical quantity in phase space is an Hermitian self adjoint operator with respect to the inner product, and that the inner product is an Hermitian inner product, *i.e.* $\langle \phi | \psi \rangle^* = \langle \psi | \phi \rangle$. The state space of the system is defined as a linear representation space of the set of fundamental commutation relations (2.21). Time evolution is determined via the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi; t\rangle = H_T |\psi; t\rangle,$$

where H_T is associated with its classical counterpart but with the variables replaced by operators.

In the classical system we had constraints $\phi_a \approx 0$, correspondingly at the quantum level gauge-invariant physical states are defined by

$$\phi_a |\psi\rangle = 0.$$

States which satisfy the above condition are called physical states. The definition of physical states should be compatible with time evolution and gauge invariance, *i.e.* physical states should remain physical under time evolution, and thus we have

$$[\phi_a, \phi_b] = \phi_a \phi_b - \phi_b \phi_a = i\hbar C_{ab}^c \phi_c, \quad (2.22)$$

$$[H_T, \phi_a] = i\hbar V_a^b \phi_b, \quad (2.23)$$

in correspondence with the associated classical statements.

The problem of enforcing these conditions raises the issue of operator ordering, as in general operators do not commute (whereas the corresponding classical

quantities do). Furthermore the coefficients, which may be functions of the z_A , may not commute with the operators. These issues have been dealt with thoroughly in other books [1, 14] and so shall not be covered here, except to say that in all situations we shall always put the operators on the right of the coefficients.

We have now (ideally) quantised a constrained (singular) classical system by means of Dirac quantisation, and it should be possible (at least in principle) to solve such a system. However as opposed to the situation that applies to regular systems, Dirac quantisation of singular systems does not allow for a straightforward path integral representation of quantum amplitudes. This problem has to do with the presence in the space of states of unphysical states, and of states related through gauge transformations. How one should proceed given this situation is far from trivial. Klauder [15] has developed a method which uses the physical projection operator within Dirac's quantisation of constrained systems. However we do not follow this route, but instead turn to alternative forms of quantisation.

2.3 Grassmann coordinates

Up until now we have only dealt with systems that can be fully described by dynamical variables that are real or complex functions of time and, hence, that belong to a commutative algebra. Such variables are quantised by means of commutators and therefore describe bosonic degrees of freedom. However in general systems this is not enough, for example a system containing fermionic degrees of freedom is very difficult to represent with a commutative algebra. Such a system is described classically by anticommuting variables belonging to a Grassmann algebra. Systems which contain both Fermionic and Bosonic degrees of freedom are described by a combination of anticommuting and commuting variables. A simple example of a system with both types of variables is a particle with spin (see chapter 5); where the position coordinates are bosonic and the spin degree of freedom is fermionic. Even for purely bosonic systems, the introduction

of Grassmann mechanics is natural when there is a gauge freedom (*i.e.* a system with constraints). This is because the introduction of ghost variables, which obey Fermi statistics, can arise in such systems. Ghost variables are a crucial part of the BRST construction, as we shall see in the next chapter.

In this section we will define a Grassmann algebra and outline properties that will be of use throughout this thesis, as well as showing how they relate to Fermi statistics.

2.3.1 Grassmann algebra

A formal description of Grassmann algebras can be found in a great many text books, for reference we have chosen [9, 13, 16].

Let $\xi^A, A = 1, \dots, N$ be a set of generators for an algebra, which anticommute

$$\{\xi^A, \xi^B\} = \xi^A \xi^B + \xi^B \xi^A = 0,$$

for all $A, B = 1, \dots, N$. Here we have used the anticommutator $\{\cdot, \cdot\}$ for the first time. Such an algebra is called a Grassmann algebra G_N with N generators. A Grassmann algebra has dimension 2^N . An obvious consequence of the defining relations for G_N is that $(\xi^A)^2 = 0$. Infinite-dimensional Grassmann algebras G_∞ are defined through the smooth limit $N \rightarrow \infty$.

A basis for G_N is given by the set of all possible monomials

$$1, \xi^1, \dots, \xi^N, \xi^1 \xi^2, \xi^1 \xi^3, \dots, \xi^{N-1} \xi^N, \dots, \xi^1 \dots \xi^N,$$

i.e. all possible combinations such that no index is repeated. A general element g of G_N can thus be written

$$g = g_0 + g_A \xi^A + g_{AB} \xi^A \xi^B + \dots + g_{A_1 \dots A_N} \xi^{A_1} \dots \xi^{A_N}, \quad (2.24)$$

where, without loss of generality, the coefficients $g_{AB}, \dots, g_{A_1 \dots A_N}$ are assumed to be completely antisymmetric in their indices.

We define the concept of Grassmann even and Grassmann odd variables as follows: A dynamical variable is an even (x^i) or odd (θ^α) variable if it can be

written with time independent generators ξ^A as

$$x^i(t) = x_0^i(t) + x_{AB}^i(t)\xi^B\xi^A + \dots, \quad (2.25)$$

$$\theta^\alpha(t) = \theta_A^\alpha(t)\xi^A + \theta_{ABC}^\alpha(t)\xi^C\xi^B\xi^A + \dots, \quad (2.26)$$

where the coefficients in x^i, θ^α are complex numbers. The series (2.25) and (2.26) are finite for a finite-dimensional Grassmann algebra because any repeated indices in the product cause it to vanish. From these definitions we get the following fundamental relations

$$\begin{aligned} \theta^\alpha\theta^\beta + \theta^\beta\theta^\alpha &= 0, & \theta^\alpha, \theta^\beta \text{ odd}, \\ \theta^\alpha x^i - x^i\theta^\alpha &= 0, & \theta^\alpha \text{ odd}, x^i \text{ even}, \\ x^i x^j - x^j x^i &= 0 & x^i, x^j \text{ even}. \end{aligned} \quad (2.27)$$

It can also be seen that two odd functions multiplied together yields an even function, an odd and an even yields an odd, whilst two evens multiplied result in an even function.

2.3.2 Differentiation and Integration

This section is a very brief outline of the calculus of Grassmann variables. If more detail is required see, for example, [13, 16].

The left derivative of the elements of a Grassmann algebra are given by

$$\frac{\partial}{\partial \xi^B} \xi^{A_1} \dots \xi^{A_p} = \delta^{A_1 B} \xi^{A_2} \dots \xi^{A_p} + \dots + (-1)^{(p-1)} \delta^{A_p B} \xi^{A_1} \dots \xi^{A_{p-1}},$$

from which it can be shown

$$\left\{ \frac{\partial}{\partial \xi^B}, \xi^A \right\} = \delta_B^A,$$

and

$$\left\{ \frac{\partial}{\partial \xi^B}, \frac{\partial}{\partial \xi^A} \right\} = 0.$$

Integration over Grassmann variables is defined as [13]

$$\int d\xi^A = 0, \quad \int d\xi^A \xi^A = 1 \text{ (not summed)}.$$

It can be shown that the $d\xi^A$ also satisfy anticommutation relations amongst themselves and with the ξ^A

$$\{d\xi^A, d\xi^B\} = 0, \quad \{d\xi^A, \xi^B\} = 0.$$

The extension to multiple integration is carried out according to a nested procedure. The extension to differentiation by x^i or θ^α can be defined along similar lines to that for ξ^A .

2.3.3 Superfunctions

The generators ξ^A were introduced so that we could view the dynamical evolution of a system as defining a trajectory in phase space. The coordinates of this space are the x^i, θ^α along with their canonically conjugate momenta $p_i = \partial\mathcal{L}/\partial x^i$, $\pi_\alpha = \partial\mathcal{L}/\partial\theta^\alpha$ in the Grassmann algebra generated by the ξ 's. However despite the fact that the ξ 's are the generators of the Grassmann algebra, they do not appear explicitly in the system that we have constructed, and thus are not really necessary. Thus we may shift the emphasis of our discussion from the points in the system (ξ^A) to the functions $(x^i, \theta^\alpha, p_j, \pi_\beta)$. In this way we view dynamical evolution as defining a map from the algebra of phase space (super)functions onto itself.

Using this philosophy we now define superfunctions over a Grassmann algebra; a Grassmann valued superfunction f is a function of the variables $x_0^i, x_{AB}^i, \dots, \theta_A^\alpha, \dots$ defined through (2.25), (2.26). f has no explicit dependence on ξ^A . A general superfunction can be expanded in powers of the odd variables θ^α as

$$f(x, \theta) = f_0(x) + f_\alpha(x)\theta^\alpha + f_{\alpha\beta}(x)\theta^\beta\theta^\alpha + \dots, \quad (2.28)$$

where the functions $f_{\alpha_1 \dots \alpha_k}(x)$ of the commuting variables x are fully antisymmetric in $\alpha_1 \dots \alpha_k$.

It can be shown [9] that the Poisson bracket is defined only for the x^i and θ^α as a whole and not for their individual components, for this reason the individual components are of no interest classically or quantum mechanically. As a consequence, one should focus at the classical level on the properties that hold as a

consequence of the basic relations (2.27) and do not depend on the expansion of the variables x^i, θ^α in terms of a basis ξ^A . In particular, one should not demand the algebra to have a particular dimension N .

Derivatives are defined by

$$\delta f = \delta \theta^\alpha \frac{\partial f}{\partial \theta^\alpha}.$$

Care is needed to keep track of minus signs when applying differentiation rules to superfunctions.

Any superfunction (from now on we shall drop the ‘super’ prefix and refer to them as functions) can be decomposed into ‘even’ and ‘odd’ components

$$f = f_E + f_O,$$

where the even (odd) component contains only the even (odd) powers of θ^α in the expansion of f . Note that in order for the function f to have a well-defined Grassmann parity, it is necessary for both f_E and f_O to have the same parity. The Grassmann parity $[f]$ of a function is defined to be 1 for an even function and (-1) for an odd function. One has

$$fg = [fg]gf,$$

for any pair of functions f and g , and where $[fg] = -1$ only if both are odd (and 1 otherwise). Alternatively, we can define a grading factor associated with f and g , such that $gr_f = 0$ if f is even and 1 if it is odd. The Grassmann parity $[f]$ is then defined as

$$[f] = (-1)^{gr_f}.$$

2.3.4 Poisson Brackets

By carrying out a similar analysis to that given in section 2.1, but this time employing Grassmann variables x^i and θ^α as well as their conjugate momenta p_i and ϖ_α respectively, we arrive at generalised Poisson brackets as follows: the

Poisson bracket of any two functions F, G is given by

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial x^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x^i} \right) + [F] \sum_\alpha \left(\frac{\partial F}{\partial \varpi_\alpha} \frac{\partial G}{\partial \theta^\alpha} + \frac{\partial F}{\partial \theta^\alpha} \frac{\partial G}{\partial \varpi_\alpha} \right), \quad (2.29)$$

where $[F]$ is the Grassmann parity of F .

The basic non vanishing Poisson brackets between the coordinates and the momenta can be calculated as

$$\begin{aligned} \{x^i, p_j\} &= -\{p_j, x^i\} = \delta_i^j, \\ \{\theta^\alpha, \varpi_\beta\} &= \{\varpi_\beta, \theta^\alpha\} = -\delta_\beta^\alpha. \end{aligned}$$

2.3.5 Grassmann variables and Fermions

The canonical quantisation procedure defined in section 2.2 is based on the correspondence

$$(\text{Poisson bracket}) \rightarrow \frac{1}{i\hbar}(\text{commutator}).$$

This description was consistent for integer spin systems, where classical variables were described by Poisson brackets, and corresponding quantum operators by a commutator. In the generalisation to Fermionic systems, whereby operator relations are described using the anticommutator, the Poisson brackets defined in section 2.1 are insufficient. Ordinary Poisson brackets do not have the same algebraic properties as anticommutators. This is where the generalisation of Poisson brackets defined in the previous section comes to the fore; classical Fermions can now be described using the Grassmann odd θ^α , and thus obeying Grassmann odd Poisson brackets, which obey the same relations as anticommutators. Classical Bosons are described using the Grassmann even variables, thus obeying standard Poisson brackets and thus normal commutation relations. The mixed case leads to a ‘normal’ Poisson bracket and thus a commutator. The above rule thus generalises naturally to encompass Poisson brackets of both Grassmann parities and the corresponding commutator/anticommutator.

When two functions F, G are necessarily odd then we shall write the anticommutator between them as $\{F, G\}$. Similarly when they are necessarily even we

shall write the commutator $[F, G]$. However when their parity is undetermined we shall write the graded commutator (which may be an anticommutator) as

$$\llbracket F, G \rrbracket = FG - [FG]GF.$$

We can likewise define a graded anticommutator (which may be a commutator) as

$$\{F, G\} = FG + [FG]GF.$$

Where we have used the convention for the $[FG] = -1$ only if both are odd, and 1 otherwise. This grading can be extended to $[FG][FH]$ *etc* as necessary.

Chapter 3

BFV-BRST Quantisation

In this chapter we outline the method of BFV-BRST quantisation in extended phase space, and show the relationship between this construction and the structure of certain classes of representations of the orthosymplectic superalgebras $osp(d, 2/2)$ and their inhomogeneous extensions. In this context the physical states (that is, the irreducible representations of the Poincaré algebra, carried on appropriate wavefunctions) appear as the resolution of the BRST complex naturally provided in the Lie superalgebraic formulation.

In the first section we will construct an extended phase space and study BRST symmetry [2, 3] which is of fundamental importance to the method of BFV quantisation. Next we shall outline the BFV quantisation of the extended phase space that was constructed in section 3.1. In doing so we shall discuss the ordering difficulties that may arise, the time evolution and the cohomology of the quantised physical space. A brief description of the algebraic aspects of the $iosp(m/n)$ algebras shall be the subject of the second part of this chapter. We shall show how, starting from Minkowski space-time and the Poincaré group we can define the generators of $iosp(d, 2/2)$. Finally we shall outline Parisi-Sourlas supersymmetry and demonstrate the relationship between BFV quantisation and the $osp(1, 1/2)$ superalgebras, before showing how this can be generalised to $iosp(d, 2/2)$.

3.1 BRST symmetry

An essential ingredient of BFV-BRST quantisation is the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [2, 3], which is manifested through the BRST operator (or charge) Ω . The central idea of the BRST theory is to substitute a fermionic rigid symmetry acting on an appropriately extended phase space for the original gauge symmetry.

The profound importance of the BRST operator was first established in quantum mechanical systems, but it has since been shown to have a natural and necessary place with classical systems as well. As such we shall, in this section, introduce the BRST operator in a classical system, and only after our study of its properties are complete shall we examine the BFV-BRST quantisation [5, 6, 7] procedure that follows from it. This may seem to be approaching things in reverse order, but in retrospect it appears more natural; BRST symmetry may have been discovered in a quantum system, but if classical phase space geometry had been extended to Grassmann variables, it could have been discovered long ago in a purely classical context. A more comprehensive description of the BRST construction, leading to BFV quantisation can be found in [9, 14].

An integral feature of BRST symmetry is the existence of ‘ghost’ variables. Ghosts were first encountered in quantum field theory as fields with the ‘wrong’ relation between spin and statistics that were necessary in addition to those fields that appeared in a corresponding classical system. The ghost fields enabled one to maintain a local description of quantum gauge theories in terms of elementary processes involving free propagation and local vertices (ghost are not unique in this ability, but other methods shall not be discussed here as they are not relevant to this thesis). They also ensured that the theory would be unitary and independent of gauge choice. It was initially thought that the ghosts were just an artifact leading to a useful representation of the measure. However with the discovery of BRST symmetry it became clear that the ghosts were of equal importance to the geometry of a system as any of the other fields (or in the case of a particle system

- operators). This point of view emerged as the logical development of the idea of gauge-invariance.

Let us assume that all phase space variables z_A are real ($z_A^* = z_A$) and have Grassmann parity $[A]$. The functions obey the Poisson bracket relations

$$\{z_A, z_B\} = C_{AB}. \quad (3.1)$$

As a consequence the algebraic structure functions obey $C_{AB}^* = -[AB]C_{BA}$. In section 2.1 we have shown how to reduce a general singular system to one with only first class constraints, and so we shall assume that this is now the case. For simplicity we shall refer to the fundamental brackets as Poisson brackets, with the understanding that these may in fact be Dirac brackets if second class constraints are present. Further, we shall assume that the regular first class constraints ϕ_α are real, have Grassmann parity $[\alpha]$, and satisfy the algebra given by (2.18) and (2.19). The Lagrange multipliers corresponding to the constraints must in turn satisfy the property

$$(\lambda^\alpha)^* = [\alpha]\lambda^\alpha,$$

as the terms $\lambda^\alpha \phi_\alpha$ which appear in the total Hamiltonian (2.13) must be Grassmann even and real. Note that we are now using the conventional symbol λ for the Lagrange multipliers, instead of the more general u or v of the earlier sections. We shall simply use H for the first-class Hamiltonian.

3.1.1 Extended phase space

In order to reveal the BRST symmetry of a constrained system the phase space is extended in two ways; firstly, to give the Lagrange multipliers the status of dynamical degrees of freedom we introduce a momentum π_α which is canonically conjugate to the corresponding λ^α . The momenta π_α have the same Grassmann parity as λ^α and are real under complex conjugation. The only non-vanishing Poisson brackets involving these conjugate degrees of freedom are

$$\{\pi_\alpha, \lambda^\beta\} = -\delta_\alpha^\beta, \quad \{\lambda^\alpha, \pi_\beta\} = [\alpha]\delta_\beta^\alpha. \quad (3.2)$$

The introduction of these new degrees of freedom leads immediately to a new set of first class constraints, namely

$$\pi_\alpha = 0.$$

These constraints are obviously first class as their Poisson brackets with all the other constraints (including other π_α) are, by definition, zero. Associated with the first class constraints π_α we have additional local gauge transformations, whose sole effect is to shift the variables λ^α by arbitrary functions, in accordance with their characterisation as Lagrange multipliers.

As the π_α are first class constraints, it is convenient to broaden our definition of the originally defined set of first class constraints ϕ_α to include the π 's. Thus we define the (new) complete set of first class constraints associated with an extended phase space as ϕ_a (roman character) where

$$\phi_a = \phi_1, \dots, \phi_K, \pi_1, \dots, \pi_K, \text{ (originally } K \text{ constraints).}$$

This new set of constraints satisfies an algebra

$$\{\phi_a, \phi_b\} = C_{ab}^c \phi_c, \tag{3.3}$$

$$\{H, \phi_a\} = V_a^b \phi_b. \tag{3.4}$$

The values of these new structure functions are easy to determine: the structure functions which arise from commutation relations involving only the original constraints (the ϕ_α) are unchanged, and the remaining functions are zero. The structure functions also obey the Jacobi identity, although in the case of a non-closed algebra this is not-trivial to prove [14].

The second extension to the phase space that is carried out in the BRST theory is to introduce conjugate pairs of ghost variables in order to account (see section 3.4) for the new degrees of freedom introduced in the phase space $(z_A, \lambda^\alpha, \pi_\alpha)$. For each first class constraint ϕ_a we introduce a pair of conjugate ghosts (η^a, ρ_a) , both of Grassmann parity opposite to that of ϕ_a . Thus if ϕ_a has Grassmann parity $[a]$, both η^a, ρ_a have opposite parity, which we shall denote $[\bar{a}]$. The only

non-vanishing Poisson brackets involving the ghosts are

$$\{\rho_a, \eta^b\} = -\delta_a^b, \quad \{\eta^a, \rho_b\} = -[a]\delta_b^a. \quad (3.5)$$

To be consistent with these fundamental brackets, the ghosts have the following properties under complex conjugation: $(\eta^a)^* = \eta^a$, $(\rho_a)^* = -[a]\rho_a$.

Thus the BRST extended phase space is parametrised by the coordinates $(z_A, \lambda^\alpha, \pi_\alpha, \eta^a, \rho_a)$, where the coordinates satisfy equations (3.1), (3.2), (3.5). The system is also subject to the constraints $\phi_a = 0$.

The final ingredient that is necessary before we introduce the BRST operator is to introduce an additive grading, known as the ghost number N_{gh} . The ghost number operator is defined by assigning the following ghost numbers to the extended phase space variables:

$$\begin{aligned} z_A, \lambda^\alpha, \pi_\alpha &: 0, \\ \eta^a &: +1, \\ \rho_a &: -1. \end{aligned}$$

The total ghost number of a quantity is obtained by adding the ghost numbers of the factors appearing in it, and is in fact the eigenvalue obtained when applying N_{gh} to a function. For example, the quantity $z_A \eta^a$ has ghost number $0 + 1 = 1$.

It can be shown [9] that the ghost number operator can be written $N_{\text{gh}} = -\eta^b \rho_b$ (classically, applying N_{gh} corresponds to multiplying the N_{gh} by its operand). Using this definition of ghost number the following example can be calculated explicitly as follows (assuming z_A is Grassmann even and thus the ghosts are odd)

$$N_{\text{gh}}(z_A \eta^a) = -\eta^b \rho_b z_A \eta^a = -z_A (-\rho_b \eta^b - 1) \eta^a = z_A \rho_b \eta^b \eta^a + z_A \eta^b = z_A \eta^b,$$

as $\eta^2 = 0$. Thus we have the claimed result.

3.1.2 The BRST operator

Within this extended phase space the BRST operator can now be defined [2, 3] through the following theorem.

Theorem: There always exists a quantity Ω , the BRST charge, which is uniquely determined up to canonical transformations by the following properties

$$\begin{aligned}
i) \quad & \Omega^* = \Omega, \\
ii) \quad & \Omega \text{ has Grassmann parity } (+1), \\
iii) \quad & \Omega \text{ has ghost number } (+1), \\
iv) \quad & \left. \frac{\partial}{\partial \eta^a} \Omega \right|_{\eta^a = \rho^a = 0} = \phi_a, \\
v) \quad & \{\Omega, \Omega\} = 0.
\end{aligned} \tag{3.6}$$

By virtue of property v), $\Omega^2 = 0$ (Ω is nilpotent).

The proof of the existence of such a quantity involves non-trivial identities that must be satisfied on the algebra, however such an Ω can be shown to always exist [14, 17].

The general form of the BRST charge (stated without proof) is

$$\Omega = \eta^a \phi_a - \frac{1}{2} [b] \eta^b \eta^a C_{ab}^c \rho_c + \eta^c \eta^b \eta^a C_{abc}^{(2)ed} \rho_d \rho_e + \dots \tag{3.7}$$

The higher-order terms are determined by the Jacobi identities of the structure functions C_{ab}^c , and the corresponding identities obeyed by successive brackets of these identities [17]. Conversely, for any given system it is possible to generate all higher-order terms of Ω through its nilpotency (starting with the first two terms of (3.7)), and thus the BRST charge can be considered the generating function for all the higher-order structure functions of the algebra.

For a closed algebra, there are no higher-order terms beyond the C_{ab}^c and thus the BRST operator Ω is completely defined by the first two terms of (3.7).

The BRST charge is of fundamental importance in (the BFV formulation of) any gauge-invariant system. The existence of the BRST charge is independent of any Hamiltonian or gauge fixing conditions, and depends only on the set of first class constraints ϕ_a and their algebra (the structure coefficients C_{ab}^c). As a consequence any two systems with identical first class constraints share an identical ghost system and BRST charge. Thus we see how the local gauge invariance of a constrained Hamiltonian system, associated with the first class constraints ϕ_α is

replaced by a global symmetry in extended phase space, generated by the BRST charge.

As an example, in the case of a closed algebra of constraints we have the BRST transformations

$$\begin{aligned}
\delta_B z_A &= \{z_A, \phi_a\} \eta^a, \\
\delta_B \lambda^\alpha &= [\alpha] \eta^\alpha, \\
\delta_B \pi_\alpha &= 0, \\
\delta_B \eta^a &= \frac{1}{2} [ac] \eta^c \eta^b C_{bc}^a, \\
\delta_B \rho_a &= -(\phi_a + [a\bar{b}] \eta^b C_{ab}^c \rho_c),
\end{aligned} \tag{3.8}$$

as compared to the local gauge transformations

$$\begin{aligned}
\delta_\xi z_A &= \{z_A, \xi^a \phi_a\} = [a] \{z_A, \phi_a\} \xi^a, \\
\delta_\xi \lambda^\alpha &= \xi^\alpha,
\end{aligned} \tag{3.9}$$

$$\delta_\xi \pi_\alpha = 0, \tag{3.10}$$

where $\xi^a \phi_a$ is the generator of local gauge transformations.

Given any first class quantity f of the phase space variables z_A , or even $(z_A, \lambda^\alpha, \pi_\beta)$, and obeying an algebra $\{f, \phi_a\} = f_a^b \phi_b$, it is possible [14] to define its BRST extension $f_B = f_B(z_A, \lambda^\alpha, \pi_\beta, \eta^a, \rho_b)$. Such an extension obeys the following rules

$$\begin{aligned}
i) \quad & f_B \text{ has same Grassmann parity } [f] \text{ as } f, \\
ii) \quad & f_B \text{ has zero ghost number,} \\
iii) \quad & f_B|_{\eta^a=\rho_a=0} = f(z_A, \lambda^\alpha, \pi_\alpha), \\
iv) \quad & \{f_B, \Omega\} = 0,
\end{aligned} \tag{3.11}$$

and can be shown [14] to have the form

$$f_B = f + [f][\bar{a}] \eta^a f_a^b \rho_b + \dots \tag{3.12}$$

Given the BRST extension f_B of a first class quantity f we may define an infinity of BRST-invariant quantities all associated with f , obtained through

$$f'_B = f_B - \{\chi, \Omega\}, \tag{3.13}$$

where χ is any function on the BRST extended phase space, with Grassmann parity $[\tilde{f}]$ and ghost number (-1) .

Comparing f'_B with the defining properties (3.11) of f_B we see that all are satisfied except *iii*). Rather $f'_B = f +$ (linear combination of first class constraints). By analogy with the definition of equivalence classes, we define the cohomology class of a BRST-invariant function f as the set of functions f_i such that f_i and f are related through equation (3.13) for some quantity χ . The cohomology classes of the BRST charge Ω are characterised by their Grassmann parity and ghost number. The BRST charge belongs to the trivial cohomology of odd Grassmann parity and ghost number $(+1)$. In fact

$$\Omega = \{N_{\text{gh}}, \Omega\} = -\{\eta^a \rho_a, \Omega\}.$$

3.1.3 Dynamics on extended phase space

The first class Hamiltonian H can be extended to a BRST form as follows

$$H_B = H + \eta^a V_a^b \rho_b + \dots,$$

where the coefficients V_a^b are those given in (2.19). All elements of the cohomology class equivalent to H_B (Grassmann even and zero ghost charge) are obtained as

$$H_{\text{eff}} = H_B - \{\mathcal{F}, \Omega\}. \quad (3.14)$$

The function \mathcal{F} is an arbitrary anti-Hermitian function on the extended phase space, with odd Grassmann parity and ghost number -1 , as such it can be written in the form

$$\mathcal{F} = \sum_{n \geq 0} \eta^{a_n} \dots \eta^{a_1} \mathcal{F}_{a_1 \dots a_n}^{(n) b_1 \dots b_{n+1}} \rho_{b_{n+1}} \dots \rho_{b_1},$$

where the coefficients $\mathcal{F}^{(n)} = \mathcal{F}^{(n)}(z_a, \lambda^\alpha, \pi_\alpha)$ have specific Grassmann parity and properties under complex conjugation.

The function \mathcal{F} is responsible for inducing a gauge fixing of the local gauge symmetries on the BRST extended system. The freedom in the choice of \mathcal{F} corresponds to the freedom in gauge fixing that is allowed by the (BFV-)BRST

formalism. Such a choice is always BRST-invariant, and thus consistent with the local gauge invariance properties, however it has to be done appropriately so as to yield an admissible gauge fixing condition of the system [14]. For some systems an admissible gauge fixing function \mathcal{F} may not exist, but this problem must be investigated on a case by case basis. By cunning choice of the function \mathcal{F} it is possible to obtain a description in which the Lagrange multipliers λ^α are set equal to specific functions, or recover the reduced phase space. Examples of suitable \mathcal{F} and how they may be used to reduce the phase space are given as they are needed in later chapters.

In section 2.1, the time evolution was obtained using a first class Hamiltonian, given as H plus an arbitrary linear combination of the first class constraints. A consequence of this is that the physical observables retained their gauge-invariant characters under time evolution. In the BRST extended phase space description, time evolution is generated by the extended Hamiltonian H_B as defined above, or more generally any member H_{eff} of the cohomology class of H_B . Moreover, with a BRST-invariant Hamiltonian, both the BRST charge Ω and ghost number N_{gh} remain constant, *i.e.* $\dot{\Omega} = 0$, $\dot{N}_{gh} = 0$.

Time evolution of any quantity f on extended phase space is now given by

$$\dot{f} = \frac{\partial f}{\partial t} + \{f, H_{eff}\}.$$

When solving the equations of motion corresponding to the Hamiltonian (3.14) specific boundary conditions must be satisfied for each pair of conjugate degrees of freedom. In the BRST extended phase space solutions describing physical systems are obtained when the BRST charge vanishes identically, since it is Ω which generates the symmetry corresponding to the local gauge symmetry (*c.f.* the conditions $\phi_\alpha = 0$ in section 2.1). Moreover, the unphysical degrees of freedom $\pi_\alpha, \eta^a, \rho_a$ must also vanish at the boundary, so that they decouple from the time evolution of the system. This in turn leads to the ghost charge N_{gh} also vanishing at the boundaries.

As we have already stated, the BRST charge Ω and ghost number N_{gh} are constant in time, and thus in order to describe a physical gauge-invariant config-

uration of the system we require

$$\Omega = 0, \quad N_{\text{gh}} = 0,$$

for all time.

It can be shown [14] that for a satisfactory description of the system, the remaining boundary conditions can be written

$$\begin{aligned} \pi_\alpha(t_i) = 0 \quad \pi_\alpha(t_f) = 0, \\ \rho_{\alpha_{(1)}}(t_i) = 0 \quad \rho_{\alpha_{(1)}}(t_f) = 0, \\ \eta^{\alpha_{(2)}}(t_i) = 0 \quad \eta^{\alpha_{(2)}}(t_f) = 0, \end{aligned} \tag{3.15}$$

where $\alpha_{(1)}$ denotes that the variables associated with the constraints ϕ_α and $\alpha_{(2)}$ are associated with the constraints $\pi_\alpha = 0$.

Thus the evolution of a system is described by solutions to the equations of motion defined through the Hamiltonian (3.14) satisfying the boundary conditions (3.15). The solutions to the equations of motion are characterised by $\Omega = 0$ and $N_{\text{gh}} = 0$.

3.2 BFV Quantisation

In a constrained system, the role of Dirac quantisation is taken by BFV quantisation of the extended phase space constructed in section 3.1. In the quantum theory the canonical variables of the extended space (including the ghosts) become operators in a linear space with a non-positive inner product. The inner product is such that the real canonical variables become Hermitian operators, whilst the imaginary ones become anti-Hermitian. The interested reader is referred to [14, 18, 9, 19] for alternative descriptions of BFV-BRST quantisation.

For the description of BFV quantisation it is convenient to revert to canonically conjugate coordinates X^M, P_M instead of the more general z_A (note that the index A runs over twice the index M). Corresponding to the Poisson brackets of the classical system we now have the (anti)commutators

$$[[X^M, P_N]] = i\hbar\delta_N^M,$$

$$\begin{aligned}
[[\pi_\alpha, \lambda^\beta]] &= -i\hbar\delta_\alpha^\beta, \\
\{[\rho_a, \eta^b]\} &= -i\hbar\delta_a^b.
\end{aligned}
\tag{3.16}$$

Quantum operators, such as the BRST operator Ω or the BRST-invariant Hamiltonian H_{eff} (the quantum analogue of (3.14)) are now defined as composite operators in the operator phase space. As a result of the Hermitian properties of the $(X^M, P_M, \lambda^\alpha, \pi_\alpha, \eta^a, \rho_a)$, both the quantum BRST operator Ω and the BRST-invariant Hamiltonian H_{eff} are also Hermitian, whilst the gauge fixing function \mathcal{F} is anti-Hermitian. In correspondence to the classical system we have

$$[H, \Omega] = 0, \quad \{\Omega, \Omega\} = 0,$$

Note that irrespective of the Grassmann properties of the operators making up the phase space the Hamiltonian is by definition Grassmann even, whilst the BRST operator is similarly Grassmann odd.

As with the quantisation of any system, we encounter the problem of ordering in the definition of composite operators. We assume that we can in fact find a normal ordering prescription of Ω that satisfies the above equations, as well as the required Hermiticity properties. Unlike in the classical case there is no guarantee that such a prescription can be found, however if it cannot be done then the system cannot be quantised in such a way as to preserve its local gauge symmetries, a situation characteristic of quantum anomalies [14].

The time evolution of the system is described by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi; t\rangle = H_{eff} |\psi; t\rangle,$$

where H_{eff} is the operator equivalent of the Hamiltonian given in (3.14)

The ghost number of an operator N_{gh} is defined as in the classical theory, with N_{gh} Hermitian, except that due to the non-commutativity of operators we require a normal ordered version. Thus N_{gh} can be represented by the operator

$$N_{gh} = \frac{i}{2} (\eta^a \rho_a + \rho_a \eta^a),$$

and obeys

$$[[N_{gh}, X^M]] = [[N_{gh}, P_M]] = [[N_{gh}, \lambda^\alpha]] = [[N_{gh}, \pi_\alpha]] = 0,$$

$$[[N_{\text{gh}}, \eta^a]] = \eta^a, \quad [[N_{\text{gh}}, \rho_a]] = -\rho_a.$$

From the equations above, it follows that if an arbitrary state $|\psi\rangle$ has ghost number ℓ (*i.e.* $N_{\text{gh}}|\psi\rangle = \ell|\psi\rangle$), then $X^A|\psi\rangle$ also has ghost number ℓ (this holds for any zero ghost number operator).

Observables in the quantised theory are operators V which have zero ghost number and are BRST-invariant, *i.e.* commute with Ω :

$$[[V, \Omega]] = 0, \quad [[V, N_{\text{gh}}]] = 0.$$

This definition is motivated by the classical theory where $\{V, \Omega\}_{\text{PB}} = 0$ implies that the quantity V has weakly vanishing brackets with the constraints. Further, two variables V and V' are considered identical if they belong to the same cohomology class, *i.e.* we can write

$$V' = V + [[\chi, \Omega]],$$

for some arbitrary χ .

This identification is compatible with the commutator operation only if we impose an extra condition on the wave function which is determined as follows. Given V and V' , defined above, and W and $W' = W + [[\chi', \Omega]]$ we require the result $[[V', W']] = [[V, W]] + [[\chi'', \Omega]]$. Expanding out this equation we get

$$[[V', W']] = [[V, W]] + [[[V + [[\chi, \Omega]], \chi'] + [[\chi, W]], \Omega],$$

but only if we consider a subset of all possible states to correspond to physical states; this subset can be worked out by observing that changing from V to V' should not change the expectation value between two physical states, and so we must have

$$\begin{aligned} \langle \psi_1 | V | \psi_2 \rangle &= \langle \psi_1 | V' | \psi_2 \rangle, \\ &= \langle \psi_1 | V | \psi_2 \rangle + \langle \psi_1 | \chi \Omega | \psi_2 \rangle + \langle \psi_1 | \Omega \chi | \psi_2 \rangle. \end{aligned}$$

Thus the above equation will be true only if the physical states are annihilated by the BRST operator, *i.e.*

$$\Omega|\psi\rangle = 0. \tag{3.17}$$

There is one further restriction on physical states which is needed to ensure equivalence with Dirac quantisation [17, 9], this is to demand that physical states are eigenfunctions of the ghost number operator

$$N_{\text{gh}}|\psi\rangle = \ell|\psi\rangle,$$

for some eigenvalue ℓ . This ensures that physical states are independent of the ghosts η^a, ρ_a . In general the appropriate physical or gauge invariant states, which coincide with those obtained through Dirac quantisation, are recovered only for a specific value of the ghost number ℓ [14]. This ℓ is the lowest or highest possible ghost number, depending on the choice of sign for the ghost number operator (this general statement is not necessarily true in the case of β -limiting, discussed in chapter 5). In fact, BRST invariant states of given ghost number are in one-to-one correspondence with Dirac's physical states for only one ghost number value.

The end result of BFV quantisation is that physical states correspond to those states which are annihilated by the BRST charge Ω and belong to the BRST cohomology of a specific ghost number ℓ . Furthermore the physical states, thus defined, which exist as a subset of the extended phase space $(z_A, \lambda^\alpha, \pi_\beta, \eta^a, \rho_b)$ are equivalent to the physical states which exist in the phase space $(X^M, P_M, \lambda^\alpha)$ arrived at using Dirac's quantisation method (section 2.2) [17, 9].

3.3 $\text{osp}(\mathbf{d}, 2/2)$

The study of the geometrical properties of phase space is known as symplectic geometry. The term symplectic is used as the geometry of the system displays a symplectic structure. Using the generalised coordinates for the original phase space $z_A = (X_M, P_M)$, where $M = 0, \dots, d-1$, then the structure of the system is completely captured by a rank-two covariant antisymmetric tensor C_{AB} such that

$$z_A, z_B = C_{AB}.$$

C_{AB} can be written explicitly as

$$C_{AB} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the complex identity matrix.

In the next section we shall demonstrate that the BRST operator Ω and the associated extended phase space can be naturally embedded into the orthosymplectic algebra $osp(d, 2/2)$, or its inhomogeneous extension $iosp(d, 2/2)$ (depending on whether we desire Lorentz or Poincaré symmetry). As a prelude to this demonstration we shall, in this section, outline the properties of the $osp(d, 2/2)$ algebra and show the construction of it from Minkowski space-time and canonical variables. More extensive descriptions of the properties of symplectic algebras and $osp(d, 2/2)$ in particular can be found in many places, for example [20, 21, 16, 22, 23, 24].

The Poincaré group, also known as the inhomogeneous Lorentz group, is the space-time symmetry group in d dimensions. This group comprises the set of space-time translations in d dimensions and the transformations which leave the metric invariant (Lorentz transformations in d dimensions). The Lorentz group is isomorphic to the pseudo-orthogonal group $O(d-1, 1)$ and so the Poincaré group can be denoted $IO(d-1, 1)$. $IO(d-1, 1)$ has the semi-direct product structure $I(d-1, 1) \oplus O(d-1, 1)$, where $I(d-1, 1)$ is the invariant sub-group of translations.

Elements of the Poincaré group can be written as a two-tuple $w = (a, \Lambda)$ made up of $a = (a^0, \dots, a^{d-1})$, the d component real vector describing translations and Λ , which is the $d \times d$ matrix of the Lorentz transformation. Multiplication between two elements w and w' of the Poincaré algebra is defined as

$$ww' = (a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda \Lambda'). \quad (3.18)$$

Which leads to an identity element $(0, 1)$, and an inverse

$$w^{-1} = (-\Lambda^{-1}a, \Lambda^{-1}). \quad (3.19)$$

The set of translations $I(d-1, 1)$ is in fact isomorphic to Minkowski space-time itself. This implies that space-time can be regarded as the left coset space

$IO(d-1,1)/O(d-1,1)$. The canonical projection map $\pi : IO(d-1,1) \rightarrow IO(d-1,1)/O(d-1,1)$ for w in $IO(d-1,1)$ is given by $\pi w = a$. The canonical action of w on a space-time point is $wx = \Lambda x + a$.

We now restrict the Lorentz transformations to those which are proper and orthochronous, *i.e.* the connected component $ISO(d-1,1)$ of $IO(d-1,1)$ is selected. We denote the universal covering group of $ISO(d-1,1)$ by the symbol P , and loosely call it the Poincaré group (technically it is the universal covering group of the connected component of the Poincaré group). By noting that $I(d-1,1)$ is its own universal covering group we see that P has the semi-direct product structure $P = I(d-1,1) \oplus L$, where L , the Lorentz group, is the universal covering group of $SO(d-1,1)$. Once again, the members of P can be written $w = (a, \Lambda)$, except now with the restriction that Λ is in L . Equations (3.18) and (3.19) still describe the group structure on P .

P is a simply connected Lie Group [16] and so is related to a Lie algebra in the standard fashion [16, 20]. The real Lie algebra of P is $iso(d-1,1)$, which decomposes into $i(d-1,1) \oplus so(d-1,1)$ as a vector space direct sum. If $M_{\mu\nu}(= -M_{\nu\mu})$ is a basis for the Lie algebra $so(d-1,1)$ of the Lorentz group, and K_λ is a basis for the invariant subalgebra $i(d-1,1)$ corresponding to the translation group, then $M_{\mu\nu}, K_\lambda$ are the basis elements of $iso(d-1,1)$, and their Lie brackets are

$$\begin{aligned} [M_{\mu\nu}, M_{\lambda\rho}] &= \eta_{\mu\lambda}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\nu\rho}M_{\mu\lambda}, \\ [M_{\mu\nu}, K_\lambda] &= \eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu, \\ [K_\mu, K_\nu] &= 0. \end{aligned} \tag{3.20}$$

In a unitary representation of the Poincaré group, the Poincaré algebra elements will be represented by anti-Hermitian operators. In physical applications we use the set of elements $J_{\mu\nu}(= -J_{\nu\mu})$ and P_λ where

$$\begin{aligned} J_{\mu\nu} &= -i\hbar M_{\mu\nu}, \\ P_\lambda &= -i\hbar K_\lambda. \end{aligned} \tag{3.21}$$

Any real linear combinations of these elements will be represented by Hermitian operators. The Lie brackets (3.20) become

$$\begin{aligned} [J_{\mu\nu}, J_{\lambda\rho}] &= -i\hbar(\eta_{\mu\lambda}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\lambda} - \eta_{\nu\lambda}J_{\mu\rho} + \eta_{\nu\rho}J_{\mu\lambda}), \\ [J_{\mu\nu}, P_\lambda] &= -i\hbar(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu), \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{3.22}$$

Strictly speaking $J_{\mu\nu}$ and P_λ are not basis elements of the real Lie algebra $iso(d-1, 1)$, but of its complexification. However, if only real linear combinations of these elements are made, then multiplying the result by i produces an element of $iso(d-1, 1)$, which maps to an element of P by exponentiation. $J_{\mu\nu}$ and P_λ will be called the generators of the Lorentz and translation groups respectively.

In order to generate $iosp(d-1, 1/2)$ we introduce a Grassmann odd 2-dimensional space, which we denote with the indices α, β, γ . We now define the symplectic rotations $K_{\alpha\beta}$ and the Grassmann odd elements $L_{\mu\alpha}, Q_\alpha$ which generate superrotations and supertranslations respectively.

If we write $J_{MN} = (J_{\mu\nu}, L_{\mu\alpha}, K_{\alpha\beta})$ and $P_M = (P_\mu, Q_\alpha)$, then the graded Lie products for the space can now be written

$$\begin{aligned} \llbracket J_{MN}, J_{PQ} \rrbracket &= i\hbar(\eta_{NQ}J_{MP} - [NP]\eta_{NP}J_{MQ} \\ &\quad - [MN][MP]\eta_{MP}J_{NQ} + [PQ][MN][MQ]\eta_{MQ}J_{NP}), \tag{3.23} \\ \llbracket J_{MN}, P_L \rrbracket &= i(\eta_{LN}P_M - [MN]\eta_{LM}P_N), \end{aligned}$$

where the generalised metric η_{MN} corresponds to the usual metric $\eta = \text{diag}(1, -1, -1, \dots)$, when $MN = \mu\nu, \mu\alpha$, and $\eta_{\alpha\beta}$ is the anti-symmetric Levi-Civita symbol. The inhomogeneous elements P_M span an Abelian, invariant sub-superalgebra, denoted $i(d-1, 1/2)$, whilst the homogeneous elements J_{MN} span the superalgebra $osp(d-1, 1/2)$.

3.4 BFV quantisation and $osp(d, 2/2)$

Most interesting relativistic theories are, in some way or another, gauge theories, and thus constrained. This is true for point particles and field theories. In the case

of point particles (the subject of this thesis) propagating over a flat Minkowski space-time of dimension d , with the Poincaré group as an affine invariant group, the relevant gauge invariance is associated with the arbitrary reparametrisation of the world-line. Furthermore, it has been shown [25, 26, 27, 28, 19, 29] that there exist natural spacetime ‘quantisation superalgebras’ which possess representations precisely mirroring the BFV-BRST construction in the case of relativistic particle systems in Minkowski space-time, and generalisations thereof for which the relevant spacetime supersymmetries are the superconformal algebra $osp(d, 2/2)$ and its inhomogeneous extension $iosp(d, 2/2)$.

In this section we outline how the quantum theory in extended phase space is formulated as a Parisi-Sourlas superspace [30] with ghost degrees of freedom viewed as negative-dimensional coordinates. The BRST operator is identified with the generator of a particular Parisi-Sourlas super-rotation in this phase space, which gives a supersymmetry which ensures both unitarity and positivity. A more thorough argument, including proofs can be seen in [19].

In order to explain how negative-dimensional coordinates can be realised by anticommuting variables, consider an extended $d + 2$ -dimensional superspace, consisting of d Minkowskian coordinates x^μ , two anticommuting coordinates $\theta^{1,2}$, along with their corresponding conjugate momenta $p_\mu = i\hbar \partial/\partial x^\mu$ and $\pi_{1,2} = \partial/\partial \theta^{1,2}$. We denote coordinates in this system by

$$x_M = (x^\mu, \theta^1, \theta^2).$$

Rotations can be generalised from d into $d + 2$ -dimensional super-rotations by replacing $x^\mu x_\mu$ with

$$x^M x_M = x^\mu x_\mu + 2\theta_1 \theta_2, \tag{3.24}$$

where the nonvanishing components of the metric tensor η_{MN} are the usual Minkowski metric $\text{diag}(-1, 1, \dots, 1)$ and $\eta_{d+1, d} = -\eta_{d+1, d} = 1$. The invariance of (3.24) determines the group of orthosymplectic super-rotations $OSp(d-1, 1/2)$, made of bosonic rotations $SO(d-1, 1)$, and fermionic rotations $Sp(2)$ that leaves $\theta^1 \theta^2$ invariant. From (3.24) we can derive the graded Lie algebra of $OSp(d-1, 1/2)$

as that generated by the operators J_{MN} which obey the first (homogeneous part) equation in (3.20). We can regard (3.20) as a grading of ordinary rotations, and as such its representations can be constructed by grading representations of ordinary rotations. This leads us to define

$$J_{MN} = X_M P_N - [MN] X_N P_M. \quad (3.25)$$

If we now consider superspace integrals of functions F that depend on the variables in the extended space in $OSp(d-1, 1/2)$ -invariant combinations, then, using partial integration and higher dimensional spherical polar co-ordinates [30, 19, 31, 21] we can prove the key result

$$\begin{aligned} \int d^D x d\theta^1 d\theta^2 F(x^\mu x_\mu + 2\theta^2 \theta^1) &= \frac{1}{\pi} \int d^{D-2} x F(x^\mu x_\mu) - \frac{1}{\pi} F(\infty), \\ &= \int d^{D-2} x F(x^\mu x_\mu), \end{aligned} \quad (3.26)$$

where we have restricted ourselves to functions F that tend toward zero at both ∞ and $-\infty$. From this we conclude that integration of the osp -invariant F over the $(d/2)$ space is equal to a $d-2$ -dimensional integral over ordinary space with a similar integrand. Thus we have shown that in $OSp(d-1, 1/2)$ -invariant integrations the two fermionic coordinates $\theta^{1,2}$ cancel out two bosonic variables.

In applications to constrained systems, the integrands in the path integral versions of (3.26) do not, in general, have such a simple $OSp(d-1, 1/2)$ form. Fortunately, it turns out that the integrands of the unphysical sectors can always be related to integrands of the form (3.26) by a simple change of variables [19].

We can now construct the phase space version of the integral (3.26). To do this we shall assume the Hamiltonian has been included as one of the constraints. We also assume we have a set of k first class constraints $\phi_a = 0$ and a corresponding number of gauge fixing conditions $\mathcal{F}^a = 0$ such that

$$\{\phi_a, \phi_b\} = 0 = \{\mathcal{F}^a, \mathcal{F}^b\}, \quad \{\phi_a, \mathcal{F}^b\} = -\delta_a^b.$$

The existence of ϕ_a and \mathcal{F}^a is guaranteed by the Darboux Theorem.

The extended phase space consists of the coordinates $(x^\mu, \lambda^a, p_\mu, \pi_a)$. This phase space is $(2d + 2k)$ -dimensional, in order to reduce it to the $2(d - k)$ -dimensional physical phase space we introduce $4k$ anti-commuting variables as negative-dimensional coordinates (one pair associated with each constraint or gauge fixing condition). We denote these variables $\eta^{1(a)}, \rho_{1(a)}, \eta^{2(a)}, \rho_{2(a)}$ with (anti)commutation relations

$$\{\eta^{1(a)}, \rho_{1(b)}\} = \{\eta^{2(a)}, \rho_{2(b)}\} = -i\delta_b^a.$$

This then completes the Parisi-Sourlas phase space. For each constraint ϕ_a there exists an eight-dimensional unphysical phase space with an equal number of Bosonic and Fermionic operators, half of which are position and the other half momentum operators. Thus for each constraint the relevant orthosymplectic supergroup must be either $Osp(2/2)$ or $OSp(1, 1/2)$. We shall now show it is the latter.

We introduce a four-dimensional Parisi-Sourlas superspace, associated with each constraint ϕ_a , with position and momentum variables

$$X^{Ma} = (\mathcal{F}^a, -\pi^a, \eta^{1(a)}, \rho^{2(a)}), \quad (3.27)$$

$$P_{Ma} = (\phi_a, \lambda_a, \rho_{1(a)}, \eta_{2(a)}), \quad (3.28)$$

where (quite obviously) M runs from 0 to 3. The commutation relations of these variables are

$$[P_{Ma}, X^{Nb}] = -\delta_a^b \delta_M^N,$$

and from (3.25) we get the following realisation of $OSp(1, 1/2)$, for each a

$$J_a^{MN} = X_a^M P_a^N - [MN] X_a^N P_a^M. \quad (3.29)$$

If we sum over all a for the generator $J^{-\theta^1}$ we get

$$J^{-\theta^1} = \eta^{1(a)} \phi_a + \eta_{2(a)} \pi^a,$$

which is identical to the BRST operator (3.7) for a system with constraints ϕ_a . In a similar fashion we can identify the anti-BRST operator (which was given in

[32]) with $J^{-\theta^2}$ and the ghost number operator with $J^{\theta^1\theta^2}$. This suggests that (3.29) is the representation of $OSp(1, 1/2)$ that ensures the equivalence between the physical phase space and the extended phase space of the Parisi-Sourlas mechanism.

Using this result one can embed the BRST operator (and corresponding anti-BRST operator), along with the generators of $Sp(2)$ into a simple superalgebra $OSp(1, 1/2)$ by adding another pair of anticommuting generators. Once this is done the embedding into the Lorentz group in d dimensions is very simple because one can extend $SO(d-1, 1) \times OSp(1, 1/2)$ to $OSp(d, 2/2)$ in a natural way [28, 25, 8, 33]. Neveu and West [27] show that the embedding goes through if one follows completely the quantisation program proposed by BFV [5, 6, 7]. Barducci *et. al.* [25] show that the extra bosonic coordinates necessary to describe the light cone structure included in $OSp(1, 1/2)$ are the proper time and the Lagrange multiplier, and that the extra generators included in the coset space $OSp(d, 2/2)/O(d, 2) \times Sp(2)$ are related to a Parisi-Sourlas symmetry.

3.4.1 Produced representation and the scalar multiplet

The method of produced and induced representations has not been used explicitly within this thesis. However the methods used in the following chapters were developed starting with a knowledge of such representations of algebras, and so we think it is appropriate to give a brief overview of the techniques from which this work evolved.

Produced representations in this context were first studied by Hartley and Cornwell [21, 20] and this line was followed by Jarvis and Tsohantjis [8, 34]. For simplicity we shall assume a scalar particle. Let Γ'_0 be a finite-dimensional representation of $SO(d, 2)$ carried by infinitely differentiable Borel functions $\phi(x)$ for any point $x = (x^\mu, x^d, x^{d+1})$, and taking values in \mathbb{C} . We shall denote the carrier space by $V'_0 = \mathbb{C}^\infty(ISO(d, 2)/SO(d, 2), \mathbb{C})$. $\Phi'_0(a, \Lambda)$ will denote the operators of the representation corresponding to an element (a, Λ) (see section 3.3) of $ISO(D, 2)$, and the representation will be denoted by the pair (Φ'_0, V'_0) . The

covariant representation of $ISO(d, 2)$ is a representation induced from the representation Γ'_0 of $SO(d, 2)$ given by

$$\Phi'_0(a, \Lambda)\phi'_0(x) = \Gamma'_0(\Lambda)\phi'_0(\Lambda^{-1}(x - a)).$$

In the case of a scalar representation $\Gamma'_0(\Lambda) = I$. This representation provides, as usual, a representation of the algebra $iso(d, 2)$ given by

$$\begin{aligned}\Phi'_0(J_{ab})\phi'_0(x) &= i \left(x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} \right) \phi'_0(x) + \Gamma'_0(J_{ab})\phi'_0(x), \\ \Phi'_0(P_a)\phi'_0(x) &= i \frac{\partial}{\partial x^a} \phi'_0(x).\end{aligned}$$

It can be shown [21] the the above representation is equivalent to the following produced representation

$$\Phi(X)\phi_0(P) = \sum \Gamma'_0((PX)_r) \phi_0(P^r),$$

where $X \in iso(d, 2)$, $(PX)_r \in U(so(d, 2))$ are to be interpreted as the $U(so(d, 2))$ -combinations of PX in $U(iso(d, 2))$ regarded as an $U(so(d, 2))$ -module and P is an element of the real vector space spanned by all combinations of the basis vectors of $U(iso(d, 2))$ which have the form

$$P^r = \prod P_0^{r_0} P_1^{r_1} \dots P_d^{r_d} P_{d+1}^{r_{d+1}}$$

for all $r = (r_0, r_1 \dots r_d, r_{d+1}) \in \mathbb{N}^{d+2}$. A general element X of $U(iso(d, 2))$ is given by

$$X = \sum A_r P^r \tag{3.30}$$

Following [20], for each element $\phi'_0 \in V'_0$, we can define a function ϕ_0 which lies in $V_0 = \text{Hom}_{U(so(d, 2))}$ and satisfies the definition of a produced algebra [8]

$$\phi_0(X) = \Phi'_0(X)\phi'_0(x_0),$$

where $x_0 \in ISO(d, 2)/SO(d, 2)$ is stable under $SO(d, 2)$. The representations (Φ'_0, V'_0) and (Φ_0, V_0) can be shown to be equivalent.

We can now proceed to state the representation (ϕ, V) of $iosp(d, 2/2)$ produced by the trivial representation of $osp(d, 2/2)$. This is precisely what should be called

a covariant scalar representation of $iosp(d, 2/2)$. The $U(iosp(d, 2/2))$ regarded as a $U(osp(d, 2/2))$ module has a basis of the form $P^r Q^s$ with P^r as in (3.30) and $Q^s = Q_1^{s_1} Q_2^{s_2}$ where $s_1, s_2 \in \{0, 1\}$ and $s \in \{0, 1\} \times \{0, 1\}$. The produced superalgebra representation is defined by

$$\begin{aligned}\Phi(X)\phi(P^r Q^s) &= \phi(P^r Q^s X), \\ \phi(AP^r Q^s) &= \Gamma(A)\phi(P^r Q^s)\end{aligned}$$

where $A \in U(osp(d, 2/2))$, $X \in iosp(d, 2/2)$ and $\phi \in V$.

We can then show that an element of V comprises the following set of four functions, defined on $ISO(d, 2)/SO(d, 2)$:

$$\phi(x), \quad (\Phi(Q_\alpha)\phi)(x) = \phi(x, \alpha), \quad (\Phi(Q_1 Q_2)\phi)(x) = \phi(x, 12).$$

Finally, the action of the operators $\Phi(X)$ for every $X \in iosp(d, 2/2)$ can be evaluated by calculating $\Phi(X)$ on these four functions [8] (the dashes have been dropped out via the equivalence mentioned above). The action for the covariant $iosp(d, 2/2)$ scalar multiplet is given by

$$\begin{aligned}\Phi(J_{ab})\phi(x) &= \Phi_0(J_{ab})\phi(x), & \Phi(P_a)\phi(x) &= \Phi_0(P_a)\phi(x), \\ \Phi(J_{ab})\phi(x, \alpha) &= \Phi_0(J_{ab})\phi(x, \alpha), & \Phi(P_a)\phi(x, \alpha) &= \Phi_0(P_a)\phi(x, \alpha), \\ \Phi(J_{ab})\phi(x, \alpha\beta) &= \Phi_0(J_{ab})\phi(x, \alpha\beta), & \Phi(P_a)\phi(x, \alpha\beta) &= \Phi_0(P_a)\phi(x, \alpha\beta), \\ \Phi(K_{\alpha\beta})\phi(x) &= 0, & \Phi(Q_\alpha)\phi(x) &= \phi(x, \alpha), \\ \Phi(K_{\alpha\beta})\phi(x, \gamma) &= \epsilon_{\alpha\gamma}\phi(x, \beta) + \epsilon_{\beta\gamma}\phi(x, \alpha), & \Phi(Q_\alpha)\phi(x, \beta) &= -\phi(x, \alpha\beta), \\ \Phi(K_{\alpha\beta})\phi(x, \beta\gamma) &= 0, & \Phi(Q_\alpha)\phi(x, \beta\gamma) &= 0, \\ \Phi(L_{a\alpha})\phi(x) &= g_{ab}x^b\phi(x, \alpha), \\ \Phi(L_{a\alpha})\phi(x, \beta) &= -g_{ab}x^b\phi(x, \alpha\beta) - i\epsilon_{\alpha\beta}\Phi_0(P_a)\phi(x), \\ \Phi(L_{a\alpha})\phi(x, \beta\gamma) &= -i\epsilon_{\beta\gamma}\Phi_0(P_a)\phi(x, \alpha).\end{aligned}\tag{3.31}$$

An indefinite inner product is given by [21, 20]

$$(\phi, \psi) = \int d^{d+2}x \Omega^{\alpha\beta} [\phi^*(x, \alpha\beta)\psi(x) - \phi^*(x)\psi(x, \alpha\beta) - \phi^*(x, \alpha)\psi(x, \beta) + \phi^*(x, \beta)\psi(x, \alpha)]$$

Under this inner product, for functions with appropriate boundary conditions, the $iso(d, 2)$ and $sp(2, \mathbb{R})$ generators are represented by Hermitian operators while the rest are anti-Hermitian.

Chapter 4

The Scalar Particle

4.1 Introduction and Main Results

In this chapter we present the first example in our program of establishing the roots of covariant quantisation of relativistic systems in the BRST complex associated with representations of classes of extended spacetime supersymmetries. The case studied here is that of the BFV-BRST quantisation of the relativistic scalar particle in d -dimensional Minkowski space, a brief description of which will be given in section 4.2. In reference [8], the supersymmetry of the scalar particle was realised using a method of produced representations. This chapter seeks to re-present and advance upon this work with the examination of the covariant BFV-BRST quantisation of the scalar particle model via a scalar representation of $iosp(d, 2/2)$. We also present a sharpening of the previous work via a covariant tensor notation for this extended spacetime supersymmetry. The relativistic scalar particle has also been studied from a BRST perspective by Govaerts [14] as well as in [27, 26], and the reader is referred to these for an alternative point of view.

Our specific results, to be elaborated in this chapter, are as follows; in section 4.3 a space of covariant superfields carrying an appropriate scalar representation of $iosp(d, 2/2)$ is introduced and studied. The generators J_{MN} of $osp(d, 2/2)$ are associated with standard configuration space coordinates and differentials

$X^M, P_N = i\partial_N$. A necessary requirement for the irreducibility of the superfield over x^M is that the mass-shell condition $P \cdot P - \mathcal{M}^2 = 0$ can be covariantly imposed at the $d + 2/2$ level. This leads to an expression for P_- in terms of the remaining generators $P_M, M \neq -$.

In Section 4.4 a ‘BRST operator’ Ω is named as one of the nilpotent odd generators of the homogeneous superalgebra (a ‘super-boost’ acting between fermionic and light cone directions), relative to a choice of ‘ghost number’ operator within the $sp(2)$ sector. A corresponding ‘gauge fixing function’ of opposite ghost number \mathcal{F} is introduced (a ‘supertranslation’ generator) leading to the physical Hamiltonian $H = \{\mathcal{F}, \Omega\}$. The cohomology of Ω is then constructed at an arbitrary ghost number. The ‘physical states’ thus defined are found to be precisely those wavefunctions which obey the conventional $(d - 1) + 1$ -dimensional Klein-Gordon equation, and moreover which have a fixed degree of homogeneity in the light cone coordinate p_+ . As the P_- constraint already dictates the evolution in the light cone time $x^- = \eta^{-+}x_+$, the analysis thus reveals that this ‘superalgebraisation’ of the BFV-BRST quantisation yields the correct scalar irreducible representation of the Poincaré algebra in $(d - 1) + 1$ -dimensions, as carried on the space of covariant solutions of the massive Klein-Gordon equation.

As this construction has been obtained purely algebraically, without the use of a physical model, it is finally the task of Section 4.5 to establish that the standard Hamiltonian BFV-BRST construction, applied to the scalar particle model [14], does indeed give rise to an identical state space structure. And this is indeed shown to be true.

4.2 Background of the Scalar Particle

Before we begin our study of the quantisation of the scalar relativistic particle we shall give a brief explanation of what exactly the free relativistic scalar particle is, and why it is important.

The scalar particle is one of the simplest possible gauge-invariant systems. The

spacetime in which it propagates is that of d -dimensional Minkowski spacetime, with a diagonal metric $g_{\mu\nu} = \text{diag}(+1, -1, \dots, -1)$. Spacetime coordinates will be denoted x^μ , where (as can be seen from the metric) $x^0 = ct$, t being the time, and c the speed of light. The remaining coordinates x^μ , $\mu = 1, 2, \dots, d-1$ are the usual space coordinates.

The world-line of the particle is parametrised through the introduction of a dimensionless parameter τ . Given this world-line parameterisation, the spacetime trajectory of the particle is described by d functions $x^\mu(\tau)$. These functions transform as vectors under spacetime Poincaré transformations, and as scalars under world-line diffeomorphisms. Under a Poincaré transformation, with a translation vector a^μ and a Lorentz transformation Λ_ν^μ we have

$$x^\mu \rightarrow x'^\mu = \Lambda_\nu^\mu x^\nu + a^\mu,$$

whereas under a world-line reparametrisation we have

$$\tau \rightarrow \tilde{\tau} = \tilde{\tau}(\tau), \quad x^\mu(\tau) \rightarrow \tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau).$$

These two classes of transformations must necessarily define symmetries of the description being adopted, in other words, the action adopted for the particle must be both a world-line and a spacetime scalar. In particular, Poincaré invariance of the action requires that the Lagrange function be independent of x^μ , and that τ derivatives of x^μ be contracted with the Minkowski metric. Thus we can write

$$S(x^\mu) = \int_{\tau_i}^{\tau_f} d\tau \mathcal{L}(\dot{x}^\mu) \quad (4.1)$$

Given the action (4.1), it is possible to derive Noether's first theorem associated with spacetime Poincaré invariance of the system, and in doing so this yields the conserved quantities

$$P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \quad (4.2)$$

and

$$M_{\mu\nu} = P_\mu x_\nu - P_\nu x_\mu, \quad (4.3)$$

where P_μ can be identified with the total covariant energy, and $M_{\mu\nu}$ with the angular momentum of the particle. P_μ is constant for classical solutions to the equations of motion, and so the Poincaré-invariant P^2 defines the invariant mass-squared of the system through $P^2 - (mc)^2 = 0$. Thus in view of (4.2), the normalisation of the action, and Lagrange function is directly related to the mass of the particle.

We have not yet imposed the condition that the action be a world-line scalar under world-line diffeomorphisms. One way to achieve this is to define the action as the total length of the world-line between its initial and final points. But how does one define this total length? There are two ways of doing this, which can be shown to be equivalent [14].

The first action we consider arises when the world-line is viewed as embedded in spacetime, and so the Minkowski metric induces a measure of length on the world-line

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \dot{x}^2 d\tau^2.$$

For a particle of mass m a possible action is therefore

$$S(x^\mu) = -mc \int_{\tau_i}^{\tau_f} d\tau \sqrt{\dot{x}^2}, \quad (4.4)$$

where c is the speed of light. This action is known as the nonlinear, or second-order action for the massive scalar particle. There is no equivalent (*i.e.* second-order) action for a massless scalar particle.

The second action for the relativistic scalar particle system may also be constructed by considering an intrinsic world-line metric $g(\tau)$, to which the world-line scalars $x^\mu(\tau)$ are coupled in an invariant manner. Taking the metric to be positive and dimensionless we can write the linear, or first-order, action as

$$S(x^\mu, g) = c \int_{\tau_i}^{\tau_f} d\tau \sqrt{g} \frac{1}{2} [g^{-1} \dot{x}^2 + \beta m^2].$$

Here β is a pure number where $\beta = 0$ corresponds to the massless particle and $\beta = 1$ the massive particle.

The above linear action can also be written in terms of the world-line vierbein (or more correctly einbein) $e(\tau)$ defined through $g(\tau) = e^2(\tau)$, as

$$S(x^\mu, e) = \frac{c}{2} \int_{\tau_i}^{\tau_f} d\tau \left[\frac{\dot{x}^2}{|e|} + \beta m^2 |e| \right]. \quad (4.5)$$

It is straightfoward to check that both (4.4) and (4.5) are invariant under world-line diffeomorphisms

$$\tau \rightarrow \tilde{\tau} = \tilde{\tau}(\tau), \quad x^\mu(\tau) \rightarrow \tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau), \quad e(\tau) \rightarrow \tilde{e}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}} e(\tau),$$

provided the interval $[\tau_i, \tau_f]$ is mapped onto itself. In fact the group of all world-line diffeomorphisms which map the interval $[\tau_i, \tau_f]$ on to itself make up the gauge group of the system.

We have now derived two forms for the action of the scalar particle. We shall now state the properties of such systems in the classical case. We choose to study the first-order action, however it can easily be shown that the two actions (4.4), (4.5) are in fact equivalent descriptions of the free relativistic scalar particle.

From equations 4.2 and 4.3 the total covariant energy and angular momentum of the particle are given by

$$P_\mu = c \frac{\dot{x}_\mu}{|e|}, \quad M_{\mu\nu} = \frac{c}{|e|} (\dot{x}_\mu x_\nu - \dot{x}_\nu x_\mu).$$

The Euler-Lagrange equations of motion can be written

$$\frac{d}{d\tau} \frac{\dot{x}_\mu}{|e|} = 0, \quad \dot{x}^2 - \beta m^2 e^2 = 0, \quad (4.6)$$

where the last equation is that for the vierbein $e(\tau)$. Note that this equation is more exactly a constraint, a consequence of the fact that the action 4.5 does not include a dependence on $\dot{e}(\tau)$.

In order to solve the equations of motion (4.6), we choose a particular gauge fixing condition. From the transformation of the vierbein under world-line diffeomorphisms, we have

$$\frac{d\tilde{\tau}}{d\tau} = \frac{e(\tau)}{\tilde{e}(\tilde{\tau})}. \quad (4.7)$$

Thus it is always possible to find a parametrisation $\tilde{\tau}(\tau)$ such that $\tilde{e}(\tilde{\tau})$ is a constant, say $\tilde{e}(\tilde{\tau}) = \lambda_0 \neq 0$, with λ_0 an arbitrary dimensionless real constant. The corresponding reparametrisation $\tilde{\tau}(\tau)$ is the solution to

$$\frac{d\tilde{\tau}}{d\tau} = \frac{e(\tau)}{\lambda_0}.$$

Note that the sign of λ_0 explicitly specifies the modular class to which $\tilde{\tau}(\tau)$ belongs. The solution to the above equation is:

$$\text{i) } \frac{d\tilde{\tau}}{d\tau} > 0 : \tilde{\tau}(\tau) = \tau_i + \frac{\kappa\Delta\tau}{\gamma} \int_{\tau_i}^{\tau} d\tau' e(\tau'), \quad \lambda_0 = \frac{\gamma}{\kappa\Delta\tau}, \quad (4.8)$$

$$\text{ii) } \frac{d\tilde{\tau}}{d\tau} < 0 : \tilde{\tau}(\tau) = \tau_f - \frac{\kappa\Delta\tau}{\gamma} \int_{\tau_i}^{\tau} d\tau' e(\tau'), \quad \lambda_0 = -\frac{\gamma}{\kappa\Delta\tau}, \quad (4.9)$$

where we have defined

$$\Delta\tau = \tau_f - \tau_i > 0, \quad \gamma = \kappa \int_{\tau_i}^{\tau_f} d\tau e(\tau), \quad (4.10)$$

and κ is a constant defined to have units of length. Thus, given any parametrisation τ of the world-line, associated with a vierbein $e(\tau)$ and a spacetime trajectory described by $x^\mu(\tau)$, there always exists a parametrisation of the same configuration, with the same end points in spacetime, such that in that parametrisation the einbein is constant. This constant value is determined by λ_0 in (4.8) or (4.9), depending on whether or not this new parametrisation belongs to the same modular class as the original parametrisation. These two possibilities are again related through the non-trivial modular transformation.

Note that the quantity γ defined in (4.10), which measures the total intrinsic length of the world-line in units of κ , is invariant under local world-line diffeomorphisms, but it changes sign under the action of the modular group.

To solve the equations of motion (4.6), let us assume that we have a world-line parametrisation such that

$$e(\tau) = \frac{1}{\kappa\Delta\tau} \gamma,$$

where γ is a real constant parameter with the units of length. By integrating (4.2), γ is seen to be invariant under local diffeomorphisms. This choice of gauge

fixing corresponds to the proper time parametrisation of the world-line. The equations of motion (4.6) can now be written

$$\ddot{x}^\mu = 0, \quad \dot{x}^\mu + \left(\frac{m\gamma}{\kappa\Delta\tau} \right) = 0,$$

together with the boundary conditions

$$x^\mu(\tau_i) = x_i^\mu, \quad x^\mu(\tau_f) = x_f^\mu, \quad \Delta x^\mu = x_f^\mu - x_i^\mu.$$

The general solution for the first-order action (and second-order as well) for the free relativistic scalar particle is therefore

$$x^\mu(\tau) = x_i^\mu + \frac{\Delta x^\mu}{\Delta\tau}(\tau - \tau_i) = x_f^\mu - \frac{\Delta x^\mu}{\Delta\tau}(\tau - \tau_f), \quad (4.11)$$

together with the constraint

$$\sqrt{(\Delta x)^2} = |\gamma|. \quad (4.12)$$

Thus we see that for a massless particle a solution exists only if $\Delta x = 0$, whilst for a massive particle equation (4.12) specifies the value of $|\gamma|$.

It is now possible to obtain the general solution for an arbitrary parametrisation. We shall not present this case here, but once again refer the reader to [14] for a more comprehensive coverage.

In the Hamiltonian formulation, we have the conjugate phase space variables x^μ, P_μ and e, π_e , where π_e is the canonically conjugate momentum to e , and P_μ also being identified with the particle energy momentum. The non-vanishing Poisson brackets are

$$\{x^\mu(\tau), P_\nu(\tau)\} = \delta_\nu^\mu, \quad \{e(\tau), \pi_e(\tau)\} = 1.$$

We also have the primary (first class) constraint

$$\pi_e = 0. \quad (4.13)$$

The canonical Hamiltonian is given by

$$H_0 = \dot{x}^\mu P_\mu + \dot{e} \pi_e - \mathcal{L} = \frac{1}{2c} |e| [P^2 + (mc)^2],$$

which in turn leads to a candidate generator for time evolution, which can be used to generate a secondary first class constraint

$$\phi = \frac{1}{2}[P^2 + (mc)^2] = 0. \quad (4.14)$$

Thus the Hamiltonian formulation associated with the action (4.1), possesses the first class constraints (4.13), (4.14), and the corresponding total Hamiltonian [14] is:

$$H_T = \frac{1}{2c}|e|[P^2 + (mc)^2] + \lambda_1\pi_e + \frac{1}{2c}\lambda_2[P^2 + (mc)^2],$$

where $\lambda_{1,2}(\tau)$ are the Lagrange multipliers associated with the first class constraints. We can now write the corresponding action in phase space, however as discussed by Govaerts [14], this action is a non-fundamental action. Upon setting $\pi_e = 0$, $\lambda = \lambda_2 + |e|$ the sector of phase space associated with the coordinate $e(\tau)$ decouples completely in a manner consistent with the local gauge invariance of the system. This first-order action then reduces to

$$S(x^\mu, P_\mu; \lambda) = \int_{\tau_i}^{\tau_f} d\tau \left[\dot{x}^\mu P_\mu - \frac{1}{2}\lambda (P^2 + (mc)^2) \right]. \quad (4.15)$$

This action now defines the fundamental Hamiltonian description of the free relativistic scalar particle in both the massless and massive case.

Finally, we briefly consider the case where the scalar particle is coupled to an external field. Once again, the interaction term must be parametrisation-invariant and so, writing the full action as $S = S_{\text{free}} + S_{\text{int}}$ some possible forms of the interaction action are:

scalar field ϕ :

$$S_{\text{int}} = \int d\tau \phi \sqrt{\dot{x}^2},$$

vector field ϕ_μ :

$$S_{\text{int}} = \int d\tau \phi_\mu \dot{x}^\mu,$$

tensor field $\phi_{\mu\nu}$:

$$S_{\text{int}} = \int d\tau \sqrt{\phi_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}.$$

An interesting example of this coupling is to a constant electromagnetic field, $\phi_\mu \sim A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu$ ($F_{\mu\nu}$ antisymmetric). In this case the Lagrangian can be written

$$\mathcal{L} = \frac{\dot{x}^2}{|e|} - \beta m^2 |e| + q F_{\mu\nu} x^\nu \dot{x}^\mu, \quad (4.16)$$

where the coupling constant q is the electric charge of the particle, and we have normalised such that $c = 1$. The Euler-Lagrange equations of motion are now

$$\frac{d}{dt} \left(\frac{\dot{x}_\mu}{|e|} \right) + q F_{\mu\nu} \dot{x}^\nu = 0,$$

along with the same constraint equation, $\dot{x}^2 + \beta m e^2 = 0$, as in the uncoupled case. Thus we see that the introduction of an external field results in an extra potential in the equations of motion for the coordinates and so yields (predictably) different dynamics.

4.3 Covariant representations of $iosp(d, 2/2)$

4.3.1 Introduction and notation

The $iosp(d, 2/2)$ superalgebra is a generalisation of $iso(d, 2)$. The supermetric η_{MN} we shall use throughout is made up of three parts; the first has block-diagonal form with the entries being the Minkowski metric tensor of $so(d-1, 1)$ with -1 occurring d times,

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, \dots, -1).$$

The second part is off-diagonal and can be written

$$\eta_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $a, b = \pm$, reflecting a choice of light cone coordinates in two additional bosonic dimensions, one spacelike and one timelike. The final part corresponds to the Grassmann odd components and is the symplectic metric tensor

$$\eta_{\alpha\beta} = \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here Greek indices α, β, \dots take values $1, 2$, whilst λ, μ, ν, \dots take values in the range $0, \dots, d-1$, whilst Latin indices a, b, c, \dots range over $\mu, \nu, \dots, +, -$. The indices M, N, \dots cover all values, and thus run over $0, \dots, d-1, +, -, 1, 2$. Note that the grading factors are 0 for Minkowski and light cone indices μ, ν, \dots, \pm and 1 for symplectic indices α, β, \dots . With these index conventions the metric thus obeys $\eta_{MN} = [MN]\eta_{NM}$.

We define $J_{MN} = -[MN]J_{NM}$ as the generators of the $osp(d, 2/2)$ superalgebra, with commutation relations as given in [35] and also equation (3.23), but repeated here for clarity

$$\begin{aligned} \llbracket J_{MN}, J_{PQ} \rrbracket &= i(\eta_{NQ}J_{MP} - [NP]\eta_{NP}J_{MQ} \\ &\quad - [MN][MP]\eta_{MP}J_{NQ} + [PQ][MN][MQ]\eta_{MQ}J_{NP}). \end{aligned} \quad (4.17)$$

The homogeneous even subalgebra is $so(d, 2) \oplus sp(2, \mathbb{R})$ with $so(d, 2)$ generated by $J_{ab} = -J_{ba}$, and $sp(2, \mathbb{R})$ by $J_{\alpha\beta} = J_{\beta\alpha}$. For clarity, we set $J_{\alpha\beta} \equiv K_{\alpha\beta} = K_{\beta\alpha}$. Likewise, the odd generators will be denoted $J_{\mu\alpha} \equiv L_{\mu\alpha}$ or $J_{\alpha\pm} \equiv L_{\alpha\pm}$. The inhomogeneous part $i(d, 2/2)$ consists of additional (super)translation generators P_M satisfying

$$\llbracket J_{MN}, P_L \rrbracket = i(\eta_{LN}P_M - [MN]\eta_{LM}P_N). \quad (4.18)$$

The $d+2$ even translations are P_μ, P_\pm acting in the $(d, 2)$ pseudo-Euclidean space, and the two odd nilpotent supertranslations are $P_\alpha \equiv Q_\alpha$.

We consider a class of covariant scalar superfield representations of $iosp(d, 2/2)$ (compare [8, 21]) acting on suitable scalar wavefunctions $\Psi(x^M)$ over $d + 2/2$ -dimensional superspace*, $(\mathcal{B} \otimes \mathcal{F} \otimes \mathcal{S})$. The $osp(d, 2/2)$ generators can be more explicitly written

$$J_{MN} = X_M P_N - [MN]X_N P_M, \quad (4.19)$$

with

$$P_N = i\partial_N = i\frac{\partial}{\partial X^N} = i\left(\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\pm}, \frac{\partial}{\partial X^\alpha}\right).$$

* \mathcal{S} denotes the superfields over $(d + 2/2)$ -dimensional superspace $(X^M) = (x^\mu, x^\pm, \theta^\alpha)$

From (4.17), (4.18) it is easy to establish the invariance of the square of the momentum operator, namely

$$\begin{aligned} [[J_{MN}, P^R P_R]] &= i(\delta_N^R P_M - [MN]\delta_M^R P_N) P_R \\ &+ i P^R [MR][RN](\eta_{RN} P_M - [MN]\eta_{RM} P_N) = 0. \end{aligned} \quad (4.20)$$

Thus the second-order Casimir is

$$P^M P_M \equiv P^\mu P_\mu + Q^\alpha Q_\alpha. \quad (4.21)$$

Similarly we get the required generalisation of the Pauli-Lubanski operator $[J_{MN}, W^{ABC} W_{CBA}] = 0$, providing a fourth-order Casimir operator; for any vector operator V_A we have

$$[J_{MN}, V_A] = i(\eta_{AM} V_N - [MN]\eta_{AN} V_M), \quad (4.22)$$

and similarly for any tensor operator V_{AB}, V_{ABC} , for example

$$\begin{aligned} [J_{MN}, V_{ABC}] &= i(\eta_{AM} V_{NBC} - [MN]\eta_{AN} V_{MBC} \\ &+ [MA][AN](\eta_{BM} V_{ANC} - [MN]\eta_{BN} V_{AMC}) \\ &+ [MA][AN][MB][BN](\eta_{CM} V_{ABN} - [MN]\eta_{CN} V_{ABM})). \end{aligned} \quad (4.23)$$

From this we can calculate

$$[J_{MN}, V^{ABC} V_{CBA}] = \eta^{AD} \eta^{BE} \eta^{CF} [J_{MN}, V_{DEF} V_{CBA}] = 0. \quad (4.24)$$

If we define $V_{ABC} = W_{ABC} = P_A J_{BC} + [BC][CA] P_C J_{AB} + [BA][AC] P_B J_{CA}$ we get the required identity. It is a relatively straight forward, yet lengthy, procedure to explicitly calculate the Pauli-Lubanski operator. In practice it will be sufficient to investigate the requirement for reducibility of the wavefunction $\Psi(x^M)$ by the mass-shell condition. We do this in the next section.

4.3.2 Reduced realisation of $iosp(d, 2/2)$ superalgebra

In order to project out irreducible representations of the full superalgebra, we require the mass-shell condition (Klein-Gordon equation): in representation terms,

a requirement for irreducibility of the $iosp(d, 2/2)$ representation,

$$(P^M \eta_{MN} P^N - \mathcal{M}^2) \Psi = 0. \quad (4.25)$$

Expanding the sum in (4.25) gives

$$(P^\mu P_\mu + P^+ P_+ + P^- P_- + Q^\alpha Q_\alpha - \mathcal{M}^2) \Psi = 0,$$

but $P^+ = \eta^{+-} P_- = P_-$ and $P^- = \eta^{-+} P_+ = P_+$, thus we can write

$$(P^\mu P_\mu + 2P_+ P_- + Q^\alpha Q_\alpha - \mathcal{M}^2) \Psi = 0.$$

Re-stating this equation to give P_- explicitly we have

$$P_- \Psi = -\frac{1}{2P_+} (P^\mu P_\mu + Q^\alpha Q_\alpha - \mathcal{M}^2) \Psi, \quad (4.26)$$

which we shall later use as the Hamiltonian, *i.e.* $P_- = H$.

The realisation of $iosp(d, 2/2)$ that we use on superfields satisfying equation (4.26) is formulated in terms of the operators $X^\mu, P_\mu = i \frac{\partial}{\partial x^\mu}$, together with $X^\alpha = \theta^\alpha, P_\alpha = Q_\alpha = i \frac{\partial}{\partial \theta^\alpha}, P_+, X_-$. In order to achieve the correct commutation relations, and thus form the generators of $iosp(d, 2/2)$ we define $X_+ = \tau I$, where I is the identity matrix (for further details on this choice see [36, 34, 8]). The non-zero commutation relations amongst these variables are thus

$$\begin{aligned} [X_\mu, P_\nu] &= -i\eta_{\mu\nu}, & \{\theta_\alpha, Q_\beta\} &= i\varepsilon_{\alpha\beta}, & [X_-, P_+] &= i, \\ [X_-, P_-] &= -iP_+^{-1}P_-, & [\theta_\alpha, P_-] &= iP_+^{-1}Q_\alpha, & [X_\mu, P_-] &= iP_+^{-1}P_\mu. \end{aligned} \quad (4.27)$$

It is clear that the $(d+2)$ -dimensional coordinates X_μ, X_\pm, X_α and momenta P_μ, P_\pm, P_α are *not* all canonically conjugate. In particular X_+ , proportional to the identity operator, simply rescales kets (at time τ), while P_- is identified with the Hamiltonian, a function of other variables (whose action also sets the rate of time development of kets via the Schrödinger equation).

We can now state the explicit forms of the generators J_{MN} of $iosp(d, 2/2)$ as:

$$\begin{aligned} J_{\mu-} &= X_\mu P_- - X_- P_\mu, & L_{\mu\alpha} &= X_\mu Q_\alpha - \theta_\alpha P_\mu, \\ J_{\mu\nu} &= X_\mu P_\nu - X_\nu P_\mu, & J_{+\mu} &= X_+ P_\mu - X_\mu P_+, \\ L_{+\alpha} &= X_+ Q_\alpha - \theta_\alpha P_+, & K_{\alpha\beta} &= \theta_\alpha Q_\beta + \theta_\beta Q_\alpha, \\ J_{+-} &= X_- P_+ - X_+ P_-, & L_{\alpha-} &= \theta_\alpha P_- - X_- Q_\alpha. \end{aligned} \quad (4.28)$$

It is straightforward to establish that these generators do indeed satisfy the commutation relations of $osp(d, 2/2)$. As an example we shall calculate

$$[[J_{\alpha-}, J_{+\mu}] = i\eta_{+-}J_{\alpha\mu}:$$

$$\begin{aligned} [[J_{\alpha-}, J_{+\mu}] &= [[\theta_\alpha P_- - X_- Q_\alpha, X_+ P_\mu - X_\mu P_+], \\ &= [[\theta_\alpha P_-, X_+ P_\mu] - [\theta_\alpha P_-, X_\mu P_+] + [X_- Q_\alpha, X_+ P_\mu] + [X_- Q_\alpha, X_\mu P_+], \\ &= 0 - \theta_\alpha [[P_-, X_\mu] P_+ + 0 - X_\mu [X_-, P_+] Q_\alpha, \\ &= -\theta_\alpha i \frac{-P_\mu}{P_+} P_+ - X_\mu i Q_\alpha, \\ &= i(\theta_\alpha P_\mu - X_\mu Q_\alpha) = iJ_{\alpha\mu}. \end{aligned}$$

We have now calculated the complete set of non-zero commutation relations between the operators X^M, P_N and have shown that they do indeed provide the correct realisation of $iosp(d, 2/2)$ on the Ψ superfields. Remarkably, *precisely* these operators will emerge as the raw material in the extended BFV-BRST Hamiltonian quantisation of the relativistic spinning particle model (Section 4.5 below). However, the algebraic setting already provides the means to complete the cohomological construction of physical states, as we now show.

4.4 Physical States

The physical states of a system can be determined by looking at the action of the BRST operator Ω and the ghost number operator N_{gh} upon arbitrary states ψ, ψ' . As is well known [14], the physical states obey the equations

$$\Omega\psi = 0, \quad \psi \neq \Omega\psi', \quad \text{and} \quad N_{\text{gh}}\psi = \ell\psi,$$

for some specific eigenvalue ℓ (corresponding to the highest or lowest ghost number), where Ω is the BRST operator, and N_{gh} is the ghost number. Therefore in order to determine the physical states we shall fix Ω and N_{gh} , and determine their actions upon an arbitrary spinor-valued superfield ψ .

Take two c -number $sp(2)$ spinors $\eta^\alpha, \eta'^\alpha$ with the following relations

$$\eta^\alpha \eta_\alpha = 0 = \eta'^\alpha \eta'_\alpha,$$

$$\eta^\alpha \eta'_\alpha = 1 = -\eta'^\alpha \eta_\alpha. \quad (4.29)$$

An example of two such spinors is

$$\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \eta' = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (4.30)$$

Below in the superfield expansions we use

$$\begin{aligned} \theta_\eta &= \eta^\alpha \theta_\alpha, & \theta'_\eta &= \eta'^\alpha \theta_\alpha, \\ \chi_\eta &= \eta^\beta \chi_\beta, & \chi'_\eta &= \eta'^\beta \chi_\beta. \end{aligned} \quad (4.31)$$

The first of these pairs of definitions leads to

$$\frac{\partial}{\partial \theta^\alpha} = \frac{\partial \theta_\eta}{\partial \theta^\alpha} \frac{\partial}{\partial \theta_\eta} + \frac{\theta'_\eta}{\partial \theta^\alpha} \frac{\partial}{\partial \theta'_\eta} = -\eta_\alpha \frac{\partial}{\partial \theta_\eta} - \eta'_\alpha \frac{\partial}{\partial \theta'_\eta},$$

and therefore

$$\eta^\alpha \frac{\partial}{\partial \theta^\alpha} = -\frac{\partial}{\partial \theta'_\eta} \quad \text{and} \quad \eta'^\alpha \frac{\partial}{\partial \theta^\alpha} = -\eta'^\alpha \eta_\alpha \frac{\partial}{\partial \theta_\eta} = \frac{\partial}{\partial \theta_\eta}. \quad (4.32)$$

Choose the BRST operator[†] and gauge fixing operators as

$$\begin{aligned} \Omega &= \eta^\alpha L_{\alpha-}, \\ \mathcal{F} &= \eta'^\alpha Q_\alpha, \end{aligned} \quad (4.33)$$

and consistently the ghost number operator $N_{\text{gh}} \equiv \eta^\alpha \eta'^\beta K_{\alpha\beta}$ satisfies

$$[N_{\text{gh}}, \Omega] = \Omega, \quad \text{and} \quad [N_{\text{gh}}, \mathcal{F}] = -\mathcal{F},$$

as required. Note that in our case

$$N_{\text{gh}} = \eta^1 \eta'^1 K_{11} + \eta^1 \eta'^2 K_{12} + \eta^2 \eta'^2 K_{22} + \eta^2 \eta'^1 K_{21} = -\frac{1}{2}(K_{11} - K_{22}).$$

4.4.1 Action of ghost number operator

From equation (4.28), we have

$$K_{\alpha\beta} = \theta_\alpha \frac{\partial}{\partial \theta^\beta} + \theta_\beta \frac{\partial}{\partial \theta^\alpha}, \quad (4.34)$$

[†]The corresponding anti-BRST operator is $\bar{\Omega} = \eta'^\alpha L_{\alpha-}$

and so

$$N_{\text{gh}} = \theta_\eta \frac{\partial}{\partial \theta_\eta} - \theta'_\eta \frac{\partial}{\partial \theta'_\eta}, \quad (4.35)$$

We can write a series expansion of an arbitrary spinor superfield ψ over $(x^\mu, x^\pm, \theta^\alpha)$ as follows

$$\psi = A + \theta^\alpha \chi_\alpha + \frac{1}{2} \theta^2 B \quad (4.36)$$

This series expansion can be re-written with respect to the spinors (4.29) as follows

$$\begin{aligned} \theta^\alpha \chi_\alpha &= \theta^\alpha \delta_\alpha^\beta \chi_\beta = \theta^\alpha (\eta^\beta \eta'_\alpha - \eta_\alpha \eta'^\beta) \chi_\beta, \\ &= \theta_\eta \chi'_\eta - \theta'_\eta \chi_\eta, \end{aligned} \quad (4.37)$$

and

$$\begin{aligned} \frac{1}{2} \theta^2 &= \frac{1}{2} \theta^\alpha \varepsilon_{\alpha\beta} \theta^\beta = \frac{1}{2} \theta^\alpha (\eta_\alpha \eta'_\beta - \eta_\beta \eta'_\alpha) \theta^\beta, \\ &= \frac{1}{2} (\theta_\eta \theta'_\eta - \theta'_\eta \theta_\eta) = \theta_\eta \theta'_\eta. \end{aligned} \quad (4.38)$$

Thus using equations (4.37) and (4.38),

$$\psi = A + \theta_\eta \chi'_\eta - \theta'_\eta \chi_\eta + \theta_\eta \theta'_\eta B. \quad (4.39)$$

From the definition of ψ and N_{gh} we have

$$\begin{aligned} N_{\text{gh}} A &= 0, & N_{\text{gh}}(\theta_\eta \chi'_\eta) &= \theta_\eta \chi'_\eta, \\ N_{\text{gh}}(\theta_\eta \theta'_\eta B) &= 0, & N_{\text{gh}}(-\theta'_\eta \chi_\eta) &= \theta'_\eta \chi_\eta, \end{aligned} \quad (4.40)$$

and so

$$N_{\text{gh}} \psi = 0 + \theta_\eta \chi'_\eta + \theta'_\eta \chi_\eta + 0 \quad (4.41)$$

In accordance with the BRST construction, we demand that

$$N_{\text{gh}} \psi = \ell \psi, \quad (4.42)$$

for some eigenvalue ℓ . We can see immediately that physical states only exist, at best, for $\ell = 0, \pm 1$, as for any other values the only ψ satisfying (4.42) is the null wavefunction.

4.4.2 Action of BRST operator

The BRST charge is defined above as $\Omega = \eta^\alpha L_{\alpha-}$. From Section 4.3.2 we can write

$$L_{\alpha-} = \theta_\alpha P_- - X_- \frac{\partial}{\partial \theta^\alpha}, \quad (4.43)$$

and so we can expand the BRST charge as follows

$$\eta^\alpha L_{\alpha-} = \eta^\alpha \theta_\alpha P_- - \eta^\alpha X_- \frac{\partial}{\partial \theta^\alpha} = \theta_\eta P_- + \frac{\partial}{\partial \theta'_\eta} X_-.$$

From (4.26),

$$P_- = \frac{-1}{2P_+} ((P^2 - \mathcal{M}^2) + Q^\alpha Q_\alpha),$$

but

$$\begin{aligned} \varepsilon^{\beta\alpha} Q_\alpha Q_\beta = Q^\alpha Q_\alpha &= (-\eta^\alpha \eta'^\beta + \eta^\beta \eta'^\alpha) \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta}, \\ &= -2 \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta}, \end{aligned}$$

therefore

$$P_- = \frac{-1}{2P_+} \left((P^2 - \mathcal{M}^2) + 2 \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta} \right). \quad (4.44)$$

The BRST operator can thus be written

$$\Omega = \eta^\alpha L_{\alpha-} = -\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} - \frac{\theta_\eta}{P_+} \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta} + \frac{\partial}{\partial \theta'_\eta} X_-. \quad (4.45)$$

By writing ψ as a series expansion to second-order (equation (4.39)), we can determine the effect of Ω on ψ . For simplicity we shall write the effect of each term of Ω on ψ separately.

1st Term:

$$-\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} \psi = -\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} A + \theta_\eta \theta'_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} \chi_\eta.$$

2nd Term:

$$-\frac{\theta_\eta}{P_+} \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta} \psi = \frac{2\theta_\eta B}{2P_+}.$$

3rd Term:

$$\frac{\partial}{\partial \theta'_\eta} X_- \psi = -X_- \chi_\eta - \theta_\eta X_- B.$$

Grouping $\Omega\psi$ with respect to coefficients of $\theta_\eta, \theta'_\eta$ and $\theta_\eta\theta'_\eta$ we can write

$$\Omega\psi = C + C_{\theta_\eta}\theta_\eta + C_{\theta'_\eta}\theta'_\eta + C_{\theta_\eta\theta'_\eta}\theta_\eta\theta'_\eta \quad (4.46)$$

where we have

$$C = -X_- \chi_\eta, \quad (4.47)$$

$$C_{\theta_\eta} = -\frac{P^2 - \mathcal{M}^2}{2P_+} A + \frac{B}{P_+} - X_- B, \quad (4.48)$$

$$C_{\theta'_\eta} = 0, \quad (4.49)$$

$$C_{\theta_\eta\theta'_\eta} = \frac{P^2 - \mathcal{M}^2}{2P_+} \chi_\eta. \quad (4.50)$$

By enforcing $\Omega\psi = 0$ we get $C_{\theta_\eta\theta'_\eta} = 0$, which by (4.50) gives the Klein-Gordon equation acting on χ_η

$$\frac{P^2 - \mathcal{M}^2}{2P_+} \chi_\eta = 0. \quad (4.51)$$

Similarly, from (4.47), we get the condition that physical states must satisfy

$$X_- \chi_\eta = 0. \quad (4.52)$$

Finally, from (4.48), and once again enforcing $\Omega\psi = 0$, we get a restriction on A and B , such that

$$\frac{1}{2}(P^2 - \mathcal{M}^2)A = (-P_+ X_- + 1) B. \quad (4.53)$$

The physical states of the system can be identified as corresponding to those that arise at ghost number $\ell = 1$ (the highest ghost number) as for this ghost number we have the spinors χ which obey the usual massive Klein-Gordon equation, from (4.26),(4.44). As well, the P_- constraint dictates the dependence of superfield components on light cone time $\tau = x^-$, as $P_- = \partial/\partial x^-$, and finally, interpreting (4.52) in the p_+ -representation (the Fourier transform of the x^+ -representation, *i.e.* $X_- \equiv X^+ = -\partial/\partial p_+$), the χ are homogeneous functions of p_+ of degree 0. Finally, given any wavefunction ψ which corresponds to a physical state, the cohomology of the BRST operator implies that the function $\psi + \Omega\psi'$ is also a solution. However, in the case of $\ell = 1$, the maximal ghost number, the space of the image of Ω , $Im\Omega$, is trivial, and so the solution ψ is unique.

4.5 BFV-BRST quantisation of the Scalar particle and $iosp(d, 2/2)$ structure

As is well known [14, 17], the BFV canonical quantisation of constrained Hamiltonian systems [5, 6, 7] uses an extended phase space description in which, to each first class constraint ϕ_a , a pair of conjugate ‘ghost’ variables (of Grassmann parity opposite to that of the constraint) is introduced. Here we follow this procedure for the scalar relativistic particle. Although our notation is adapted to the massive case, $\mathcal{M} > 0$, as would follow from the second-order action corresponding to extremisation of the proper length of the particle world-line, an analysis of the *fundamental* Hamiltonian description of the first-order action [14] leads to an equivalent picture (with an additional mass parameter $\mu \neq 0$ supplanting m in appropriate equations, and permitting $m \rightarrow 0$ as a smooth limit).

In either case, for the scalar (or spinning particle) the primary first class constraint is the mass-shell condition $\phi_1 = (P^2 - \mathcal{M}^2)$, where $P^2 = P^\mu \eta_{\mu\nu} P^\nu$. Including the Lagrange multiplier λ as an additional dynamical variable leads to a second constraint, reflecting conservation of its conjugate momentum π_λ . The quantum formulation should be consistent with the equations of motion and gauge fixing at the classical level, as such two restrictions are necessary so as to arrive at the particle quantisation corresponding with the superalgebraic prescription of Section 4.3. Firstly, we choose below to work in the class [26, 37, 38, 9] $\dot{\lambda} = 0$. Moreover, we take gauge fixing to be with respect to orientation preserving (mapping τ_i to τ_i and τ_f to τ_f) or orientation reversing (mapping τ_i to τ_f and τ_f to τ_i) gauge transformations. Thus the gauge group of the system under consideration is the group of all world line diffeomorphisms mapping the interval $[\tau_i, \tau_f]$ onto itself. Its connected component, corresponding to the local gauge invariance of the system is the set of all orientation preserving reparametrisations in τ leaving the end points fixed. Its disconnected component, corresponding to global gauge transformations, is the set of orientation reversing reparametrisations in τ , exchanging the endpoints. Thus the modular group of the particle is non-trivial

[14] and is isomorphic to the abelian group of order two \mathbb{Z}_2 . Secondly, we take $\phi_2 = \lambda\pi_\lambda$ as the other first class constraint (rather than $\phi_2 = \pi_\lambda$ used in the standard construction). The question of the regularity of a system containing this constraint has been brought to our attention [39]. In chapter 2, we defined regularity in this sense to correspond to all the constraints being independent. The constraint ϕ_2 , as we have defined it, is still an independent constraint as we have restricted the Lagrange multiplier λ to the half line \mathbb{R}^+ , and chosen to work in the class $\dot{\lambda} = 0$. Thus the difference between the two forms of the constraint ϕ_2 is irrelevant. If it were not for this restriction then the system would be irregular.

4.5.1 BFV extended state space and wavefunctions

The BFV extended phase space [14] for the BRST quantisation of the scalar relativistic particle is taken to comprise the following canonical variables:

$$x^\mu(\tau), p_\mu(\tau), \lambda(\tau), \pi_\lambda(\tau), \eta^a, \rho_a, \quad a = 1, 2. \quad (4.54)$$

$x^\mu(\tau), p_\mu(\tau)$ are Grassmann even, λ is the Grassmann even Lagrange multiplier corresponding to the even first class constraint ϕ_1 , π_λ is the momentum conjugate to λ (which forms the constraint ϕ_2). η^1, ρ_1 and η^2, ρ_2 are the Grassmann odd conjugate pairs of ghosts corresponding to the constraints ϕ_1 and ϕ_2 respectively. We proceed directly to the quantised version by introducing the Schrödinger representation. The operators X^μ, P_ν corresponding to the coordinates x^μ, p_ν , acting on suitable sets of wavefunctions over x^μ , and on the half line $\lambda > 0$. The restriction for λ to be positive is consistent with the differential representation of $osp(d, 2/2)$ (see section 6.2 where it arise naturally via the identify $\lambda = e^\phi$, for some coordinate ϕ). The Hermitian ghosts η^a, ρ_b are represented as usual either on a 4-dimensional indefinite inner product space $|\sigma\sigma'\rangle$, $\sigma, \sigma' = \pm$, or here, in order to match with Section 4.3, in terms of suitable Grassmann variables acting on superfields. The non zero commutation relations amongst (4.54) read (repeated in full for clarity):

$$[X_\mu, P_\nu] = -i\eta_{\mu\nu}, \quad [\lambda, \pi_\lambda] = i, \quad \{\eta^i, \rho_j\} = -i\delta_j^i, \quad (4.55)$$

from which we can establish

$$\{\phi_a, \phi_b\} = 0. \quad (4.56)$$

The Hermiticity conditions imposed on the above operators read

$$\begin{aligned} X_\mu^\dagger &= X_\mu, & P_\mu^\dagger &= P_\mu, & \mu, \nu &= 0, \dots, d-1, \\ \lambda^\dagger &= \lambda, & \pi_\lambda^\dagger &= \pi_\lambda, \\ (\eta^i)^\dagger &= \eta^i, & (\rho_a)^\dagger &= -(-1)^{a+1} \rho_a, & a &= 1, 2. \end{aligned} \quad (4.57)$$

The ghost number operator N_{gh} is defined by

$$N_{\text{gh}} = \frac{i}{2} \sum_{a=1}^2 (\eta^a \rho_a - (-1)^{(a-1)} \rho_a \eta^a). \quad (4.58)$$

The canonical BRST operator[†] is given by

$$\Omega = \eta^1 \phi_1 + \eta^2 \phi_2. \quad (4.59)$$

The gauge fixing operator [26] \mathcal{F} which will lead to the appropriate effective Hamiltonian is given by:

$$\mathcal{F} = -\frac{1}{2} \lambda \rho_1, \quad (4.60)$$

and thus the Hamiltonian can be written

$$H = i [\mathcal{F}, \Omega] = -\frac{1}{2} \lambda (\eta^2 \rho_1 + \phi_1), \quad (4.61)$$

which is of course BRST-invariant.

Consider the following canonical transformations on the classical dynamical variables of the extended phase space [25]

$$\eta'^a = \lambda \eta^a, \quad (4.62)$$

$$\rho'_a = \frac{1}{\lambda} \rho_a,$$

$$\pi'_\lambda = \pi_\lambda + (-\eta^1 \rho_1 + \rho_2 \eta^2), \quad (4.63)$$

with the remainder invariant. At the same time we relabel the coordinates $p_+ = \lambda^{-1}$ and $x_- = \lambda \pi_\lambda$. At the quantum level the corresponding BRST operator

$$(\Omega' = \eta'^1 \phi_1 + \eta'^2 \phi_2),$$

[†]The criteria for the construction and nilpotency of the corresponding anti-BRST operator $\bar{\Omega}$ have been given in [32]

can be written as

$$\Omega' = \lambda\eta^1\phi_1 + \eta^2 : \lambda\phi_2 : - \lambda\eta^2\eta^1\rho_1, \quad (4.64)$$

where the symmetric ordering

$$: \lambda\phi_2 := \frac{1}{2}(\lambda\phi_2 + \phi_2\lambda) = \lambda\phi_2 - \frac{1}{2}\lambda,$$

has been introduced.

It is also convenient to define [25] the operators θ_α, Q_α , ($\alpha = 1, 2$) by

$$Q_{1,2} = \frac{i}{2\sqrt{2}}(2\eta^2 \pm \rho_1), \quad \theta_{1,2} = \frac{i}{\sqrt{2}}(\pm\rho_2 - 2\eta^1), \quad (4.65)$$

which obey the relation $\{Q_\alpha, \theta_\beta\} = i\varepsilon_{\alpha\beta}$. In terms of these variables we attain the following simple forms for the BRST, gauge fixing and Hamiltonian operators.

$$\begin{aligned} \Omega' &= \frac{-i}{\sqrt{2}} (: \lambda\phi_2 : (Q_1 + Q_2) + (\theta_1 + \theta_2)H), \\ \mathcal{F}' &= -\frac{1}{2}\rho_{1(1)} = -\frac{1}{\sqrt{2}}(Q_1 - Q_2), \\ H' &= i[\mathcal{F}', \Omega'] = -\frac{\lambda}{2}(P^\mu P_\mu + Q^\alpha Q_\alpha - \mathcal{M}^2) \equiv H. \end{aligned} \quad (4.66)$$

The forms (4.66) can now be shown to be identical to the previously given algebraically defined expressions for these quantities (4.26), (4.33). The raw material (4.27) also appears in this construction, as can be easily observed by (4.55), and by identifying $P_+ = \lambda^{-1}, X_- =: \lambda\phi_2 :$ and the BRST operator $\Omega' = \eta^\alpha L_{\alpha-}$. Moreover, the realisation of $iosp(d, 2/2)$ can be done as in (4.28). In particular, the evaluation of the BRST cohomology performed in Section 4.4 above, gives precisely the correct identification of physical state wavefunctions for the scalar particle model of this section.

4.6 Conclusions

In this chapter we have considered in detail the canonical BFV-BRST quantisation of the scalar relativistic particle, and its relationship to the extended quantisation supersymmetry superalgebra $iosp(d, 2/2)$. In a previous paper on

the quantisation of the scalar relativistic particle [8] a covariant scalar produced module of the quantisation superalgebra was identified with the extended state space of the particle quantisation. In this chapter an alternative method was adopted, whereby the quantisation was examined via a scalar representation of $iosp(d, 2/2)$. We also developed a covariant tensor notation for the extended spacetime supersymmetry.

In particular, we introduced an appropriate scalar representation of $iosp(d, 2/2)$, the homogeneous generators J_{MN} of which were defined in terms of standard configuration space coordinates and differentials X^M, P_N . We then named a BRST operator Ω as one of the nilpotent odd generators of the homogeneous superalgebra, along with a corresponding gauge fixing fermion \mathcal{F} , and Hamiltonian H . The physical states of the system were found to be wavefunctions which obey the conventional $(d - 1) + 1$ Klein-Gordon equation. This analysis revealed that our superalgebraisation of the BFV-BRST quantisation of the scalar particle yielded the correct scalar irreducible representation of the Poincaré algebra as carried on the space of covariant solutions of the massive Klein-Gordon equation. Finally, as the above program was carried out in a purely algebraic way, section 4.5 was necessary to establish that the standard Hamiltonian BFV-BRST construction for the scalar particle does indeed give rise to an identical state space structure.

The scalar particle is the first example given in this thesis of approaching covariant quantisation models with gauge symmetries via a cohomological realisation of the appropriate space of irreducible representations of physical states through the construction of the correct BRST complex. We shall continue with other examples in chapters 5 and 6 before discussing the overall program and future directions in 8.

Chapter 5

The Spinning Particle

5.1 Introduction and Main Results

In chapter 4 we examined the covariant BFV-BRST quantisation of the scalar particle via a representation of $iosp(d, 2/2)$. This chapter follows a more extended process for the spinning particle [40, 41, 42, 43, 44, 45, 46, 47, 48, 49], one of the simplest examples of a supersymmetric system. A brief explanation of the properties of the spinning particle will be given in section 5.2.

In Section 5.3, a space of covariant spinor superfields carrying an appropriate spin representation of $iosp(d, 2/2)$ is introduced, and its structure studied. The generators J_{MN} of $osp(d, 2/2)$ have orbital and spin components, associated respectively with standard configuration space coordinates and differentials $X^M, P_M = i\partial_M$, and an extended (graded) Clifford algebra with generators Γ_N entailing both Fermionic and Bosonic oscillators. The mass shell condition $P \cdot P - \mathcal{M}^2 = 0$ factorises, allowing the Dirac condition $\Gamma \cdot \partial - \mathcal{M} = 0$ to be covariantly imposed at the $d + 2/2$ - dimensional level, effecting a decomposition of the representation space. At the same time, the Dirac wavefunctions split into upper and lower components, so that the $iosp(d, 2/2)$ algebra is effectively realised on $2^{d/2}$ -dimensional Dirac spinors (over x^M , and subject to a certain differential constraint on P_- , deriving from the mass-shell condition).

In Section 5.4 a ‘BRST operator’ Ω is named as one of the nilpotent odd gener-

ators of the homogeneous superalgebra (a ‘super-boost’ acting between Fermionic and light cone directions), relative to a choice of ‘ghost number’ operator within the $sp(2)$ sector. Correspondingly a ‘gauge fixing Fermion’ \mathcal{F} of opposite ghost number is identified (a ‘supertranslation’ generator), and physical Hamiltonian $H = -\{\mathcal{F}, \Omega\}$. Finally, the cohomology of Ω is constructed at arbitrary ghost number. It is found that the ‘physical states’ thus defined are precisely those wavefunctions which obey the conventional $(d-1)+1$ -dimensional Dirac equation, and moreover which have a fixed degree of homogeneity in the light cone coordinate p_+ . Given that the P_- constraint already dictates the evolution of the Dirac spinors in the light cone time $x^- = \eta^{-+}x_+$, the analysis thus reveals that this ‘superalgebraisation’ of the BFV-BRST quantisation yields the correct spin- $\frac{1}{2}$ irreducible representation of the Poincaré algebra in $(d-1)+1$ -dimensions, as carried on the space of covariant solutions of the massive Dirac equation.

As this construction has been obtained purely algebraically, without the use of a physical model, Section 5.5 establishes that the standard Hamiltonian BFV-BRST construction, applied to the spinning particle model[40, 41, 42, 43, 46, 47, 48, 49], does indeed give rise to an identical state space structure. The only proviso on this statement turns out to be that the model’s extended phase space should formally be modified by a contraction, or ‘ β - limit’ [14] in order to identify the appropriate sector of the full phase space (for details see Section 5.5.2).

5.2 Background to the spinning particle

A relativistic spinning particle is essentially a free relativistic scalar particle (see chapter 4) which also has a spin degree of freedom (*i.e.* possesses the property of spin). Spin is essentially of a quantum nature, and therefore a classical description of a spinning particle may seem slightly nonsensical. However there are several reasons why one may wish to consider the properties of such a system; the properties of a classical spinning system are interesting by themselves as one can develop a better intuition about a particular limit of quantum theory; in dual and

field theories involving Fermionic variables, the classical limit is interesting as it leads to insights into Fermionic strings and the interaction of gravitational fields with matter; and most importantly for this thesis, by studying such a system we gain a deeper understanding of quantisation techniques and the role that the $iosp(d, 2/2)$ algebra plays in the process.

The earliest attempt to provide a classical formulation of a particle with spin involved a spinning top and was carried out by Frenkel in 1926. However his [50] and subsequent attempts [51] to quantise such a particle were unsuccessful. It was not until the almost simultaneous suggestions by Berezin and Marinov [44] and Casalbuoni [41] that the dynamics of a classical spinning particle were best represented by Grassmann algebras that quantisation of such a particle was achieved. Since then the techniques of supergravity have been applied to the problem, and in fact the spinning particle is identical to supergravity in one dimension (with position x^μ replaced by the field ϕ^μ). In general, Bose-Fermi systems, or supersymmetric systems, of which the spinning particle is perhaps the simplest, are used in many theories, for example supergravity, superstrings and M-theory.

For the description of the classical spinning particle we follow [45], but note that this treatment is similar to that of [52] and [42].

The spinning particle is described by its position $x^\mu(\tau)$ together with an additional set of Grassmann odd variables ξ_μ which commute with x^μ but anticommute with themselves

$$\xi_\mu \xi_\nu + \xi_\nu \xi_\mu = 0,$$

for any μ, ν .

There have been several, essentially similar, forms given for the action of the spinning particle. All of these have been based on Grassmann coordinates for the spinning section and give identical equations of motion. Proceeding as for the linear action in section 4.2, we construct a reparametrisation-invariant action which includes the vierbein field e . For a massless particle, the simplest

Lagrangian [45] which satisfies the necessary conditions (section (4.5)) is

$$\mathcal{L} = \frac{1}{2} \left(\frac{\dot{x}^2}{e} - i\xi \cdot \dot{\xi} \right).$$

This Lagrangian transforms as a total derivative under reparametrisation if ξ^μ transforms as a scalar like x^μ , and the corresponding action is invariant. Due to the time component of the coordinate ξ^μ there is a possibility that negative norm states may appear in the physical spectrum. In order to decouple these states we require an additional invariance under local supergauge transformations [45]. This is achieved by introducing a Fermionic counterpart, χ , to the vierbein field e , and writing the Lagrangian as follows

$$\mathcal{L} = \frac{1}{2} \left(\frac{\dot{x}^2}{e} - i\xi \dot{\xi} - \frac{i}{e} \chi \dot{x} \cdot \xi \right).$$

In constructing the massive case we need to aim at the mass-shell condition, leading to the Dirac equation (see (5.15)) and the Klein-Gordon equation. This is achieved by introducing an additional (Minkowski scalar) Grassmann variable ξ_5 which carries a mass in the constraint. We also need to include the term em^2 which carried mass in the scalar case (see (4.5)). Thus the total action for the massive case can be written [45, 11]

$$S = \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau \left\{ \frac{\dot{x}^2}{e} + em^2 - i(\xi \dot{\xi} - \xi_5 \dot{\xi}_5) - i\chi \left(\frac{\xi \cdot \dot{x}}{e} - m\xi_5 \right) \right\}. \quad (5.1)$$

The Euler-Lagrange equations of motion of the above system are

$$\begin{aligned} \frac{d}{d\tau} \left(\frac{2\dot{x}_\mu}{e} - \frac{i\chi\xi_\mu}{e} \right) &= 0, \\ \dot{\xi}_\mu - \frac{\chi\dot{x}_\mu}{2e} &= 0, \\ 2\dot{\xi}_5 - m\chi &= 0 \\ \frac{\dot{x}\xi}{e} - m\xi_5 &= 0, \\ -\frac{\dot{x}^2}{e^2} + m^2 + \frac{i\chi\dot{x}\dot{\xi}}{e^2} &= 0. \end{aligned} \quad (5.2)$$

The canonical momenta for the system are calculated by using

$$\pi_M = \frac{\partial \mathcal{L}}{\partial \dot{q}^M},$$

where q^M stands for any of the coordinates, thus the momenta can be written

$$\begin{aligned} p_\mu &= \frac{1}{e}(\dot{x}_\mu - \frac{1}{2}i\chi\xi_\mu), \\ \pi_e &= 0, \\ \pi_\mu^\xi &= \frac{i}{2}\xi_\mu, \\ \pi_\chi &= 0, \\ \pi_5 &= -\frac{i}{2}\xi_5. \end{aligned}$$

The last four of these equations form the primary first class constraints. The canonical Hamiltonian is given by

$$\begin{aligned} H_0 &= \dot{x}^\mu p_\mu + \dot{\xi}_\mu \pi_\mu^\xi + \dot{\xi}_5 \pi_5 - \mathcal{L}, \\ &= \frac{1}{2}e(p^2 - m^2) + \frac{i}{2}\chi(\xi \cdot p - m\xi_5). \end{aligned} \quad (5.3)$$

Unfortunately, the system as it is currently formulated contains neither a maximal number of first class constraints, nor is free of second class constraints. However through the process of gauge reduction using Dirac brackets and partial gauge fixing for the first class primary constraints we can arrive at the fundamental action for the system [11]. In this form we are left with the equations of motion

$$\ddot{x}^\mu = 0, \quad \dot{\xi}_\mu = 0, \quad \dot{\xi}_5 = 0,$$

the (primary first class) constraints

$$p^2 - m^2 = 0, \quad \xi \cdot p - m\xi_5 = 0,$$

and the Dirac brackets

$$\{x^\mu, p_\mu\}_D = \delta_\nu^\mu, \quad \{\xi_\mu, \xi_\nu\}_D = \eta_{\mu\nu}, \quad \{\xi_5, \xi_5\}_D = 1.$$

Finally, an alternative form of this action [11], which involves the ‘essential’ constraints, namely those which upon quantisation lead to the Klein-Gordon and Dirac equations, can be written

$$S = \int_{\tau_i}^{\tau_f} d\tau \left[\dot{x}^\mu p_\mu - \frac{i}{2}(\xi^\mu \dot{\xi}_\mu - \xi_5 \dot{\xi}_5) - \frac{\lambda_1}{2}(p^2 - m^2) - \frac{i\lambda_2}{2}(\xi^\mu p_\mu - m\xi_5) \right]. \quad (5.4)$$

Comparing this equation with (4.15) we see the obvious similarities: once again we have a correspondence between the pair e, χ and the Lagrange multipliers λ_1, λ_2 .

Upon quantisation the Grassmann coordinates ξ_μ, ξ_5 convert to the usual elements of a Clifford algebra γ_μ, γ_5 where

$$\xi_\mu = \frac{1}{\sqrt{2}}\gamma_5\gamma_\mu, \quad \xi_5 = \frac{1}{\sqrt{2}}\gamma_5.$$

5.3 Covariant representations of $iosp(d, 2/2)$

5.3.1 Introduction and notation

The supermetric η_{MN} we shall use in this chapter is identical to that given in 4.3, but will be repeated here for clarity. η_{MN} is made up of three parts; the first has block-diagonal form with the entries being the Minkowski metric tensor of $so(d, 1)$ with -1 occurring d times, $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, \dots, -1)$. The second part, corresponding to the \pm coordinates, consists of a 2×2 off-diagonal unit matrix. The final part corresponds to the Grassmann odd components and is the symplectic metric tensor $\varepsilon_{\alpha\beta}$. The conventions used for indices are also identical to the scalar particle chapter.

Although in this chapter we are dealing with the spinning particle the commutation relations for the generators of the $osp(d, 2/2)$ superalgebra remain unchanged [35]

$$\begin{aligned} \llbracket J_{MN}, J_{PQ} \rrbracket &= i(\eta_{NQ}J_{MP} - [NP]\eta_{NP}J_{MQ} - [MN][MP]\eta_{MP}J_{NQ} \\ &\quad + [PQ][MN][MQ]\eta_{MQ}J_{NP}). \end{aligned} \quad (5.5)$$

However now the J_{MN} can be split into two parts

$$J_{MN} = J_{MN}^L + J_{MN}^S, \quad (5.6)$$

where J^L is the orbital part, and is identical to the J defined in chapter 4. Thus J^L can be written $J_{MN}^L = X_M P_N - [MN]X_N P_M$ (see equation 4.19). J_{MN}^S is the

new spin component of the generators J_{MN} , and is composed of the Grassmann odd generators, we can write J^S as

$$J_{MN}^S = \frac{1}{4} \llbracket \Gamma_M, \Gamma_N \rrbracket, \quad (5.7)$$

where Γ_M, Γ_N are generalised Dirac matrices. Of course both J^L and J^S fulfill the $osp(d, 2/2)$ algebra.

The graded Clifford algebra with generators Γ_N , acting on the space* $(\mathcal{B} \otimes \mathcal{F})$, is defined through

$$\{\Gamma_M, \Gamma_N\} = \Gamma_M \Gamma_N + [MN] \Gamma_N \Gamma_M = 2[MN] \eta_{MN}, \quad (5.8)$$

(if $M \neq N$ then we can write $\Gamma_M \Gamma_N = -[MN] \Gamma_N \Gamma_M$). Writing the Γ in compact form as $\Gamma_M = (\Gamma_\mu, \Gamma_+, \Gamma_-, \Gamma_\alpha)^T$, we have

$$\begin{aligned} \Gamma_\mu &= 1 \otimes \hat{\gamma}_\mu \otimes 1, \\ \Gamma_\pm &= 1 \otimes \hat{\gamma}_\pm \otimes 1, \\ \Gamma_\alpha &= \zeta_\alpha \otimes \hat{\gamma}_5 \otimes (-1)^z, \end{aligned} \quad (5.9)$$

where

$$\begin{aligned} \hat{\gamma}_\mu &= \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix}, \quad \hat{\gamma}_+ = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \hat{\gamma}_- &= \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{\gamma}_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}, \end{aligned} \quad (5.10)$$

$(-1)^z$ is the parity operator for the θ s and is such that $(-1)^z \theta_\alpha = -\theta_\alpha$, and $(-1)^z \theta_\alpha \theta_\beta = \theta_\alpha \theta_\beta$, and is defined such that $z \equiv \theta^\alpha \partial_\alpha$, γ_5 is defined such that $\gamma_5^2 = \kappa_5 (= \pm 1)$. Note that we have implicitly limited ourselves to a space with d even, as is necessary for the definition of $\hat{\gamma}_5$.

The definition of $J_{\alpha\beta}^S$ leads to

$$4J_{\alpha\beta}^S = \llbracket \Gamma_\alpha, \Gamma_\beta \rrbracket = \zeta_\alpha \gamma_5 \zeta_\beta \gamma_5 (-1)^{2z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \zeta_\beta \gamma_5 \zeta_\alpha \gamma_5 (-1)^{2z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

* \mathcal{B} is the Bosonic part which carries the representation of ζ_α (see below), while the Fermionic \mathcal{F} carries the representation of the Dirac algebra γ_μ, γ_\pm and γ_5

$$= \kappa_5 \{\zeta_\alpha, \zeta_\beta\} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad (5.11)$$

moreover $\{\Gamma_\alpha, \Gamma_\beta\} = -2\varepsilon_{\alpha\beta}$ and so we take

$$[\zeta_\alpha, \zeta_\beta] = -2\kappa_5 \varepsilon_{\alpha\beta}. \quad (5.12)$$

In appendix A we provide a realisation of ζ_α (with $\kappa_5 = 1$) in terms of a pair of Bosonic oscillators with indefinite metric.

From equations (5.5), (4.18) it is easy to establish the invariance of the square of the momentum operator, namely

$$\begin{aligned} [[J_{MN}, P^R P_R]] &= i (\delta_N^R P_M - [MN] \delta_M^R P_N) P_R \\ &+ i P^R [MR] [RN] (\eta_{RN} P_M - [MN] \eta_{RM} P_N) = 0. \end{aligned} \quad (5.13)$$

Thus the second order Casimir is $P^2 = P^M P_M \equiv P^\mu P_\mu + Q^\alpha Q_\alpha$. We can similarly derive the Pauli-Lubanski operator, as was done in section 4.3.

5.3.2 Dirac condition and reduced realisation of $iosp(d, 2/2)$ superalgebra

In order to project out irreducible representations of the full superalgebra, we require the mass-shell condition (Klein-Gordon equation): in representation terms, a requirement for reducibility of the $iosp(d, 2/2)$ representation.

$$(P^M \eta_{MN} P^N - \mathcal{M}^2) \Psi = 0. \quad (5.14)$$

However, using the Clifford algebra just defined we have $[[J_{MN}, \Gamma^L P_L]] = 0$ and so we can covariantly impose the stronger Dirac condition,

$$\begin{aligned} 0 &= (P^M \eta_{MN} P^N - \mathcal{M}^2), \\ &= (P^M \Gamma_M + \mathcal{M}) (P^M \Gamma_M - \mathcal{M}), \end{aligned}$$

i.e. , taking for example the positive root,

$$(P^M \Gamma_M - \mathcal{M}) \Psi = 0. \quad (5.15)$$

We now construct the explicit forms of the generators J_{MN} of $iosp(d, 2/2)$ within this decomposition of the full space. Expanding the sum in (5.15) gives

$$(\Gamma^\mu P_\mu + \Gamma^+ P_+ + \Gamma^- P_- + \Gamma^\alpha Q_\alpha - \mathcal{M}) \Psi = 0,$$

or, in the explicit form, writing Ψ as a two component array $\Psi = \begin{pmatrix} \psi \\ \sqrt{2}\phi \end{pmatrix}$, gives

$$\begin{pmatrix} \gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha - \mathcal{M} & \sqrt{2} P_+ \\ \sqrt{2} P_- & -[\gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha + \mathcal{M}] \end{pmatrix} \begin{pmatrix} \psi \\ \sqrt{2}\phi \end{pmatrix} = 0.$$

From this, we get the rather useful expression

$$\phi = -\frac{1}{2P_+} (\gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha - \mathcal{M}) \psi, \quad (5.16)$$

and so P_- can be written

$$P_- \psi = -\frac{1}{2P_+} (\gamma^\mu P_\mu + \zeta^\alpha (-1)^z \gamma_5 Q_\alpha + \mathcal{M}) (\gamma^\nu P_\nu + \zeta^\beta (-1)^z \gamma_5 Q_\beta - \mathcal{M}) \psi. \quad (5.17)$$

Simplifying this equation yields

$$\begin{aligned} P_- \psi &= -\frac{1}{2P_+} (P^2 + Q_\alpha \varepsilon^{\beta\alpha} Q_\beta \kappa_5 - \mathcal{M}^2) \psi, \\ &= -\frac{1}{2P_+} (P^2 + Q^\alpha Q_\alpha \kappa_5 - \mathcal{M}^2) \psi, \end{aligned} \quad (5.18)$$

which we shall later use as the Hamiltonian. This equation is basically the Klein-Gordon equation (see equation (5.14)) of the BfV quantised spinning relativistic particle model which will carry the representation.

We are now in a position to explicitly determine the generators of $osp(d, 2/2)$. We show below the process for calculating three of the more difficult terms and then state without proof all other terms, with the understanding that the same process is repeated for each:

We have $J_{\alpha-} = J_{\alpha-}^L + J_{\alpha-}^S$, where $J_{\alpha-}^L = X_\alpha P_- - X_- P_\alpha$ and

$$\begin{aligned} J_{\alpha-}^S &= \frac{1}{4} [\Gamma_\alpha, \Gamma_-] = \frac{1}{2} \Gamma_\alpha \Gamma_-, \\ &= \frac{1}{2} \begin{pmatrix} \zeta_\alpha \gamma_5 (-1)^z & 0 \\ 0 & -\zeta_\alpha \gamma_5 (-1)^z \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

$$= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\zeta_\alpha\gamma_5(-1)^z \\ 0 & 0 \end{pmatrix}, \quad (5.19)$$

and so

$$\begin{aligned} J_{\alpha-}\Psi &= \begin{pmatrix} X_\alpha P_- - X_- P_\alpha & \frac{1}{\sqrt{2}}\zeta_\alpha\gamma_5(-1)^z \\ 0 & -X_\alpha P_- + X_- P_\alpha \end{pmatrix} \begin{pmatrix} \psi \\ \sqrt{2}\phi \end{pmatrix}, \\ &= \begin{pmatrix} (X_\alpha P_- - X_- P_\alpha)\psi + \zeta_\alpha\gamma_5(-1)^z\phi \\ -(X_\alpha P_- - X_- P_\alpha)\sqrt{2}\phi \end{pmatrix}. \end{aligned}$$

Substituting in from equation (5.16) gives

$$\begin{aligned} J_{\alpha-} &= X_\alpha P_- - X_- P_\alpha \\ &\quad - \frac{\zeta_\alpha}{2P_+} (\gamma_5(-1)^z \gamma \cdot P + \gamma_5(-1)^z \zeta^\beta (-1)^z \gamma_5 P_\beta - \gamma_5(-1)^z \mathcal{M}), \\ &= X_\alpha P_- - X_- P_\alpha - \zeta_\alpha \zeta_-, \end{aligned} \quad (5.20)$$

where

$$\zeta_- = \frac{1}{2P_+} (\gamma_5(-1)^z \gamma \cdot P + \zeta^\beta \kappa_5 P_\beta - \gamma_5(-1)^z \mathcal{M}). \quad (5.21)$$

For $J_{\mu-}$ we have

$$J_{\mu-}^S = \frac{1}{4} [\Gamma_\mu, \Gamma_-] = \frac{1}{2} \Gamma_\mu \Gamma_- = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\gamma_\mu \\ 0 & 0 \end{pmatrix},$$

therefore

$$J_{\mu-}\Psi = \begin{pmatrix} (X_\mu P_- - X_- P_\mu)\psi + \gamma_\mu\phi \\ (X_\mu P_- - X_- P_\mu)\sqrt{2}\phi \end{pmatrix},$$

thus by substituting in equation (5.16), we get

$$J_{\mu-} = (X_\mu P_- - X_- P_\mu) - \frac{\gamma_\mu\gamma_5(-1)^z}{2P_+\kappa_5} \{ \gamma_5(-1)^z \gamma \cdot P + \zeta^\alpha P_\alpha \kappa_5 - (-1)^z \gamma_5 \mathcal{M} \}.$$

This leads, upon substitution of equation 5.21, to

$$J_{\mu-} = (X_\mu P_- - X_- P_\mu) - \zeta_\mu \zeta_-,$$

where we have defined

$$\zeta_\mu = \frac{\gamma_\mu\gamma_5(-1)^z}{\kappa_5}. \quad (5.22)$$

And lastly, for $J_{\mu\nu}$ we have

$$\begin{aligned} J_{\mu\nu} &= J_{\mu\nu}^L + J_{\mu\nu}^S, \\ &= (X_\mu P_\nu - X_\nu P_\mu) \mathbf{1} + \frac{1}{4} [\Gamma_\mu, \Gamma_\nu], \\ &= X_\mu P_\nu - X_\nu P_\mu + \frac{1}{4} [\gamma_\mu, \gamma_\nu], \end{aligned}$$

but by using equation 5.22 we get that

$$\begin{aligned} [\zeta_\mu, \zeta_\nu] &= \frac{\gamma_\mu \gamma_5 \gamma_\nu \gamma_5 (-1)^2}{\kappa_5^2} - \frac{\gamma_\nu \gamma_5 \gamma_\mu \gamma_5 (-1)^2}{\kappa_5^2}, \\ &= - \left(\frac{\gamma_\mu \gamma_\nu \gamma_5^2 (-1)^2}{\kappa_5^2} - \frac{\gamma_\nu \gamma_\mu \gamma_5^2 (-1)^2}{\kappa_5^2} \right), \\ &= -\frac{1}{\kappa_5} [\gamma_\mu, \gamma_\nu], \end{aligned}$$

and so

$$J_{\mu\nu} = (X_\mu P_\nu - X_\nu P_\mu) - \frac{\kappa_5}{4} [\zeta_\mu, \zeta_\nu]. \quad (5.23)$$

In a similar fashion, we can also show

$$\{\zeta_\mu, \zeta_\nu\} = -\frac{1}{\kappa_5} \{\gamma_\mu, \gamma_\nu\} = -2\kappa_5 \eta_{\mu\nu}. \quad (5.24)$$

The complete set of generators can thus be written:

$$\begin{aligned} J_{\mu-} &= X_\mu P_- - X_- P_\mu - \zeta_\mu \zeta_-, & J_{\mu\alpha} &= X_\mu P_\alpha - X_\alpha P_\mu + \frac{\kappa_5}{2} \zeta_\mu \zeta_\alpha, \\ J_{\mu\nu} &= X_\mu P_\nu - X_\nu P_\mu - \frac{\kappa_5}{4} [\zeta_\mu, \zeta_\nu], & J_{+\mu} &= X_+ P_\mu - X_\mu P_+, \\ J_{+\alpha} &= X_+ P_\alpha - X_\alpha P_+, & J_{\alpha\beta} &= X_\alpha P_\beta + X_\beta P_\alpha + \frac{\kappa_5}{4} \{\zeta_\alpha, \zeta_\beta\}, \\ J_{+-} &= X_- P_+ - X_+ P_- - \frac{1}{2}, & J_{\alpha-} &= X_\alpha P_- - X_- P_\alpha - \zeta_\alpha \zeta_-. \end{aligned} \quad (5.25)$$

The non-zero commutation relations between ζ_- and the remaining operators for $iosp(d, 2/2)$ can be calculated, once again we shall present three as examples:

$$\begin{aligned} \{\zeta_\mu, \zeta_-\} &= \left\{ \left[\frac{\gamma_\mu \gamma_5 (-1)^z}{\kappa_5}, \frac{1}{2P_+} (\gamma_5 (-1)^z \gamma \cdot P + \kappa_5 \zeta^\beta P_\beta - \gamma_5 (-1)^z \mathcal{M}) \right] \right\}, \\ &= \frac{1}{2\kappa_5 P_+} \{[\gamma_\mu \gamma_5, \gamma_5 \gamma^\nu P_\nu] - \frac{\mathcal{M}}{\kappa_5} \{[\gamma_\mu \gamma_5, \gamma_5]\}, \\ &= \frac{P_\mu}{P_+}. \end{aligned} \quad (5.26)$$

$$\begin{aligned}
\{\zeta_-, \zeta_-\} &= \left(\frac{1}{2P_+} (\gamma_5(-1)^z \gamma \cdot P + \kappa_5 \zeta^\beta P_\beta - \gamma_5(-1)^z \mathcal{M}) \right)^2, \\
&= \frac{1}{4P_+^2} ((\gamma_5(-1)^z \gamma \cdot P)^2 + (\zeta^\beta \kappa_5 P_\beta)^2 + (\gamma_5(-1)^z \mathcal{M})^2 \\
&\quad + \text{cross terms which cancel}), \\
&= -\frac{\kappa_5}{4P_+} (\gamma \cdot P)^2 - \frac{1}{4P_+} \{\zeta^\beta, \zeta^\alpha\} P_\beta P_\alpha + \frac{2\kappa_5}{4P_+} \mathcal{M}^2, \\
&= -\frac{\kappa_5}{2P_+} (P^2 + \varepsilon^{\beta\alpha} P_\alpha P_\beta + \mathcal{M}^2), \\
&= \frac{P_-}{P_+}.
\end{aligned} \tag{5.27}$$

Defining $\zeta_- = \zeta_-^*/P_+$ we get

$$\begin{aligned}
[X_-, \zeta_-] &= \left[X_-, \frac{\zeta_-^*}{P_+} \right], \\
&= -\frac{1}{P_+^2} [X_-, P_+] \zeta_-^*, \\
&= -\frac{i}{P_+^2} \zeta_-^* = -i \frac{\zeta_-}{P_+}.
\end{aligned} \tag{5.28}$$

The full set of relations are thus

$$\begin{aligned}
\{\zeta_\mu, \zeta_-\} &= \frac{P_\mu}{P_+}, \quad \{\gamma_5, \zeta_-\} = -\frac{\kappa_5(-1)^z}{P_+} \mathcal{M}, \quad \{\zeta_-, \zeta_-\} = \frac{P_-}{P_+}, \quad [X_-, \zeta_-] = -i \frac{\zeta_-}{P_+}, \\
[X_\mu, \zeta_-] &= -i \frac{\kappa_5}{2P_+} \zeta_\mu, \quad \{X_\alpha, \zeta_-\} = -i \frac{\kappa_5 \zeta_\beta}{2P_+}, \quad [\zeta_-, \zeta_\alpha] = -\frac{Q_\alpha}{P_+}.
\end{aligned} \tag{5.29}$$

In summary, the realisation of $iosp(d, 2/2)$ that we use is formulated in terms of the operators $X^\mu, P_\mu = i \frac{\partial}{\partial x^\mu}, \gamma_\mu, \gamma_5$, together with $X^\alpha = \theta^\alpha, P_\alpha = Q_\alpha = i \frac{\partial}{\partial \theta^\alpha}, \zeta_\alpha, \zeta_-$, and $X_+ = \tau I, P_- = H, P_+, X_-$. The non-zero commutation relations amongst these variables are

$$\begin{aligned}
[X_\mu, P_\nu] &= -i\eta_{\mu\nu}, \quad \{\theta_\alpha, Q_\beta\} = i\varepsilon_{\alpha\beta}, \quad [X_-, P_+] = i, \\
[X_-, P_-] &= -iP_+^{-1}P_-, \quad [\theta_\alpha, P_-] = iP_+^{-1}Q_\alpha, \quad [X_\mu, P_-] = iP_+^{-1}P_\mu,
\end{aligned} \tag{5.30}$$

and

$$\{\zeta_\mu, \zeta_\nu\} = -2\kappa_5 \eta_{\mu\nu}, \quad [\zeta_\alpha, \zeta_\beta] = -2\kappa_5 \varepsilon_{\alpha\beta}. \tag{5.31}$$

Note in the above that X_+ and P_- are no longer canonically conjugate when acting on the ψ part of the superfield.

We have now calculated the complete set of non-zero commutation relations between the operators $X^M, P_N, \zeta_M, \gamma_5$ and have shown that they do indeed provide the correct realisation of $iosp(d, 2/2)$ on the ψ superfields. Remarkably, *precisely* these operators will emerge as the raw material in the extended BFV-BRST Hamiltonian quantisation of the relativistic spinning particle model (Section 5.5 below). However, the algebraic setting already provides the means to complete the cohomological construction of physical states, as we now show.

5.4 Physical States

Following the construction of chapter 4 we now look at the action of the BRST operator Ω and ghost number operator N_{gh} upon arbitrary states ψ, ψ' . The physical states obey the equations $\Omega\psi = 0$, $(\psi \neq \Omega\psi')$, and $N_{\text{gh}}\psi = \ell\psi$, for some eigenvalue ℓ . Therefore in order to determine the physical states we shall fix Ω and N_{gh} , and determine their actions upon an arbitrary spinor-valued superfield ψ . In order to do this we again take two c -number $sp(2)$ spinors $\eta^\alpha, \eta'^\alpha$ with the following relations

$$\begin{aligned}\eta^\alpha \eta_\alpha &= 0 = \eta'^\alpha \eta'_\alpha, \\ \eta^\alpha \eta'_\alpha &= 1 = -\eta'^\alpha \eta_\alpha.\end{aligned}\tag{5.32}$$

We shall use the same spinors as in chapter 4, *i.e.*

$$\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \eta' = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.\tag{5.33}$$

Defining the terms

$$\begin{aligned}\theta_\eta &= \eta^\alpha \theta_\alpha, & \theta'_\eta &= \eta'^\alpha \theta_\alpha, \\ \chi_\eta &= \eta^\beta \chi_\beta, & \chi'_\eta &= \eta'^\beta \chi_\beta,\end{aligned}\tag{5.34}$$

we again have

$$\eta^\alpha \frac{\partial}{\partial \theta^\alpha} = -\frac{\partial}{\partial \theta'_\eta} \quad \text{and} \quad \eta'^\alpha \frac{\partial}{\partial \theta^\alpha} = -\eta'^\alpha \eta_\alpha \frac{\partial}{\partial \theta_\eta} = \frac{\partial}{\partial \theta_\eta}.\tag{5.35}$$

Choosing the BRST operator[†] and gauge fixing operators as

$$\begin{aligned}\Omega &= \eta^\alpha L_{\alpha-}, \\ \mathcal{F} &= \eta'^\alpha P_\alpha,\end{aligned}\tag{5.36}$$

and consistently the ghost number operator $N_{\text{gh}} \equiv \eta^\alpha \eta'^\beta K_{\alpha\beta}$ satisfies

$$[N_{\text{gh}}, \Omega] = \Omega, \quad \text{and} \quad [N_{\text{gh}}, \mathcal{F}] = -\mathcal{F},$$

as required.

5.4.1 Action of ghost number operator

$K_{\alpha\beta}$ can be written $K_{\alpha\beta} = K_{\alpha\beta}^S + K_{\alpha\beta}^L$ where the two parts denote the Bosonic and Fermionic (spin and orbital) contributions respectively. This leads to N_{gh} having two parts as well;

$$K_{\alpha\beta}^L = \theta_\alpha \frac{\partial}{\partial \theta^\beta} + \theta_\beta \frac{\partial}{\partial \theta^\alpha},\tag{5.37}$$

therefore

$$N_{\text{gh}}^L = \theta_\eta \frac{\partial}{\partial \theta_\eta} - \theta'_\eta \frac{\partial}{\partial \theta'_\eta},\tag{5.38}$$

and for the Bosonic sector

$$K_{\alpha\beta}^S = \frac{1}{4} [\Gamma_\alpha, \Gamma_\beta] = \frac{1}{4} \{\zeta_\alpha, \zeta_\beta\} \kappa_5,\tag{5.39}$$

therefore

$$\begin{aligned}N_{\text{gh}}^S &= \frac{\kappa_5}{4} ((\eta \cdot \zeta)(\eta' \cdot \zeta) + (\eta' \cdot \zeta)(\eta \cdot \zeta)), \\ &= \frac{\kappa_5}{2} (\eta \cdot \zeta)(\eta' \cdot \zeta) + \frac{\kappa_5}{4} = \frac{\kappa_5}{2} (\eta' \cdot \zeta)(\eta \cdot \zeta) - \frac{\kappa_5}{4}.\end{aligned}\tag{5.40}$$

As seen in appendix A, we can write a series expansion of an arbitrary spinor superfield ψ over $(x^\mu, x^\pm, \theta^\alpha)$ in an occupation number basis in the indefinite metric space acted on by ζ_α ,

$$\begin{aligned}\psi &= \sum_{m,n=0}^{\infty} \psi^{(m,n)} |m, n\rangle = \sum_{m,n=0}^{\infty} \left(A^{(m,n)} + \theta^\gamma \chi^{(m,n)}_\gamma + \frac{1}{2} \theta^2 B^{(m,n)} \right) |m, n\rangle, \\ &= A + \theta^\gamma \chi_\gamma + \frac{1}{2} \theta^2 B.\end{aligned}\tag{5.41}$$

[†]The corresponding anti-BRST operator is $\bar{\Omega} = \eta'^\alpha L_{\alpha-}$

We can re-write this series expansion with respect to the spinors (5.32) as follows

$$\begin{aligned}\theta^\alpha \chi_\alpha &= \theta^\alpha \delta_\alpha^\beta \chi_\beta = \theta^\alpha (\eta^\beta \eta'_\alpha - \eta_\alpha \eta'^\beta) \chi_\beta, \\ &= \theta_\eta \chi'_\eta - \theta'_\eta \chi_\eta,\end{aligned}\tag{5.42}$$

and

$$\begin{aligned}\frac{1}{2}\theta^2 &= \frac{1}{2}\theta^\alpha \varepsilon_{\alpha\beta} \theta^\beta = \frac{1}{2}\theta^\alpha (\eta_\alpha \eta'_\beta - \eta_\beta \eta'_\alpha) \theta^\beta, \\ &= \frac{1}{2}(\theta_\eta \theta'_\eta - \theta'_\eta \theta_\eta) = \theta_\eta \theta'_\eta.\end{aligned}\tag{5.43}$$

Thus using equations (5.42) and (5.43),

$$\psi^{(m,n)} = A^{(m,n)} + \theta_\eta \chi_\eta'^{(m,n)} - \theta'_\eta \chi_\eta^{(m,n)} + \theta_\eta \theta'_\eta B^{(m,n)}.\tag{5.44}$$

In what follows, the occupation number labels in the Bosonic space will be suppressed, whereas the structure of the explicit superfield expansion will be needed.

Thus for example $A \equiv \sum_{m,n=0}^\infty A^{(m,n)} |m, n\rangle$, as indicated above in (5.41).

It can be easily seen that

$$\begin{aligned}N_{\text{gh}}^L A &= 0, & N_{\text{gh}}^L (\theta_\eta \chi'_\eta) &= \theta_\eta \chi'_\eta, \\ N_{\text{gh}}^L (\theta_\eta \theta'_\eta B) &= 0, & N_{\text{gh}}^L (-\theta'_\eta \chi_\eta) &= \theta'_\eta \chi_\eta,\end{aligned}\tag{5.45}$$

and so

$$\begin{aligned}N_{\text{gh}} \psi &= \frac{1}{2} \left(\kappa_5 (\eta \cdot \zeta) (\eta' \cdot \zeta) + \frac{\kappa_5}{2} \right) A + \frac{1}{2} \theta_\eta \left(\kappa_5 (\eta \cdot \zeta) (\eta' \cdot \zeta) + \frac{\kappa_5 + 4}{2} \right) \chi'_\eta \\ &\quad - \frac{1}{2} \theta'_\eta \left(\kappa_5 (\eta \cdot \zeta) (\eta' \cdot \zeta) + \frac{\kappa_5 - 4}{2} \right) \chi_\eta \\ &\quad + \frac{1}{2} \theta_\eta \theta'_\eta \left(\kappa_5 (\eta \cdot \zeta) (\eta' \cdot \zeta) + \frac{\kappa_5}{2} \right) B.\end{aligned}\tag{5.46}$$

We demand that $N_{\text{gh}} \psi = \ell \psi$ for some eigenvalue ℓ , therefore we can write

$$\kappa_5 \left(\frac{1}{2} (\eta \cdot \zeta) (\eta' \cdot \zeta) + \frac{1}{4} \right) A = \ell A,$$

i.e.

$$\kappa_5 (\eta \cdot \zeta) (\eta' \cdot \zeta) A = \left(\frac{4\ell - \kappa_5}{2} \right) A.\tag{5.47}$$

Similarly

$$\kappa_5(\eta \cdot \zeta)(\eta' \cdot \zeta)\chi'_\eta = \left(\frac{4\ell - \kappa_5 - 4}{2} \right) \chi'_\eta, \quad (5.48)$$

$$\kappa_5(\eta \cdot \zeta)(\eta' \cdot \zeta)\chi_\eta = \left(\frac{4\ell - \kappa_5 + 4}{2} \right) \chi_\eta, \quad (5.49)$$

$$\kappa_5(\eta \cdot \zeta)(\eta' \cdot \zeta)B = \left(\frac{4\ell - \kappa_5}{2} \right) B. \quad (5.50)$$

In appendix A the diagonalisation of $(\eta \cdot \zeta)(\eta' \cdot \zeta)$ is carried out explicitly in the occupation number basis $\{|m, n\rangle\}$. Below we assume that suitable eigenstates can be found, and explore the consequences for the cohomology of the BRST operator at generic ghost number ℓ .

5.4.2 Action of BRST operator

The BRST charge is defined above as $\Omega = \eta^\alpha L_{\alpha-}$ and from Section 5.3.2 we can write

$$L_{\alpha-} = \theta_\alpha P_- - X_- \frac{\partial}{\partial \theta^\alpha} - \zeta_\alpha \zeta_-. \quad (5.51)$$

We have previously, in (5.21), defined $\zeta_- = \frac{1}{2P_+} (D_5(-1)^z + \kappa_5 \zeta^\beta P_\beta)$, where $D_5 = \gamma_5(\gamma \cdot P - \mathcal{M})$ is the Dirac operator multiplied by γ_5 . Consequently

$$\begin{aligned} \eta^\alpha \zeta_\alpha \zeta_- &= \frac{\eta^\alpha \zeta_\alpha}{2P_+} (D_5(-1)^z + \kappa_5 \zeta^\beta P_\beta), \\ &= \frac{\eta^\alpha \zeta_\alpha (-1)^z D_5}{2P_+} + \kappa_5 \frac{(\eta \cdot \zeta) \left(\zeta^\beta \frac{\partial}{\partial \theta^\beta} \right)}{2P_+}. \end{aligned} \quad (5.52)$$

The second part of equation (5.52) can be further expanded as follows

$$\zeta^\beta \frac{\partial}{\partial \theta^\beta} = \zeta_\gamma \frac{\partial}{\partial \theta^\beta} \varepsilon^{\beta\gamma} = \zeta_\gamma \frac{\partial}{\partial \theta^\beta} (-\eta^\beta \eta'^\gamma + \eta^\gamma \eta'^\beta), \quad (5.53)$$

which uses the identity $\varepsilon^{\beta\alpha} = (-\eta^\beta \eta'^\alpha + \eta^\alpha \eta'^\beta)$. Therefore

$$\frac{(\eta \cdot \zeta) \left(\zeta^\beta \frac{\partial}{\partial \theta^\beta} \right)}{2P_+} = \frac{1}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) \frac{\partial}{\partial \theta'_\eta} + \frac{1}{2P_+} (\eta \cdot \zeta)^2 \frac{\partial}{\partial \theta_\eta}, \quad (5.54)$$

and so we can write

$$\eta^\alpha \zeta_\alpha \zeta_- = \frac{\eta^\alpha \zeta_\alpha (-1)^z D_5}{2P_+} + \frac{\kappa_5}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) \frac{\partial}{\partial \theta'_\eta} + \frac{\kappa_5}{2P_+} (\eta \cdot \zeta)^2 \frac{\partial}{\partial \theta_\eta}. \quad (5.55)$$

In a similar fashion we can expand the Fermionic part of $L_{\alpha-}$ as follows

$$\eta^\alpha L_{\alpha-}^L = \eta^\alpha \theta_\alpha P_- - \eta^\alpha X_- \frac{\partial}{\partial \theta^\alpha} = \theta_\eta P_- + \frac{\partial}{\partial \theta'_\eta} X_- ,$$

where

$$P_- = \frac{-1}{2P_+} ((P^2 - \mathcal{M}^2) + Q^\alpha Q_\alpha) .$$

Using the identity

$$\begin{aligned} \varepsilon^{\beta\alpha} Q_\alpha Q_\beta = Q^\alpha Q_\alpha &= (-\eta^\alpha \eta'^\beta + \eta^\beta \eta'^\alpha) \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} , \\ &= -2 \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta} , \end{aligned}$$

thus allows as to write

$$P_- = \frac{-1}{2P_+} \left((P^2 - \mathcal{M}^2) + 2 \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta} \right) . \quad (5.56)$$

The BRST operator can thus be written

$$\begin{aligned} \Omega = \eta^\alpha L_{\alpha-} &= -\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} - \frac{\theta_\eta}{P_+} \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta} + \frac{\partial}{\partial \theta'_\eta} X_- \\ &\quad - \frac{\eta^\alpha \zeta_\alpha (-1)^z D_5}{2P_+} - \kappa_5 \frac{(\eta \cdot \zeta)(\eta' \cdot \zeta)}{2P_+} \frac{\partial}{\partial \theta'_\eta} - \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} \frac{\partial}{\partial \theta_\eta} . \end{aligned} \quad (5.57)$$

By writing ψ as a series expansion to second-order (equation (5.44)), we can determine the effect of Ω on ψ . For simplicity we shall write the effect of each term of Ω on ψ separately.

1st Term:

$$-\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} \psi = -\theta_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} A + \theta_\eta \theta'_\eta \frac{P^2 - \mathcal{M}^2}{2P_+} \chi_\eta .$$

2nd Term:

$$-\frac{\theta_\eta}{P_+} \frac{\partial}{\partial \theta_\eta} \frac{\partial}{\partial \theta'_\eta} \psi = \frac{2\theta_\eta B}{2P_+} .$$

3rd Term:

$$\frac{\partial}{\partial \theta'_\eta} X_- \psi = -X_- \chi_\eta - \theta_\eta X_- B .$$

4th Term:

$$\begin{aligned} -\frac{\eta^\alpha \zeta_\alpha (-1)^z D_5}{2P_+} \psi &= -(\eta \cdot \zeta) \frac{D_5}{2P_+} A + \theta_\eta (\eta \cdot \zeta) \frac{D_5}{2P_+} \chi'_\eta \\ &\quad - \theta'_\eta (\eta \cdot \zeta) \frac{D_5}{2P_+} \chi_\eta - \theta_\eta \theta'_\eta (\eta \cdot \zeta) \frac{D_5}{2P_+} B . \end{aligned}$$

5th Term:

$$-\kappa_5 \frac{(\eta \cdot \zeta)(\eta' \cdot \zeta)}{2P_+} \frac{\partial}{\partial \theta'_\eta} \psi = \frac{\kappa_5}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) \chi_\eta + \kappa_5 \frac{\theta_\eta}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) B.$$

6th Term:

$$-\kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} \frac{\partial}{\partial \theta_\eta} \psi = -\frac{\kappa_5}{2P_+} (\eta \cdot \zeta)^2 \chi'_\eta - \kappa_5 \theta'_\eta - \frac{\theta'_\eta}{2P_+} (\eta \cdot \zeta)^2 B.$$

Grouping $\Omega\psi$ with respect to coefficients of $\theta_\eta, \theta'_\eta$ and $\theta_\eta \theta'_\eta$ we can write

$$\Omega\psi = C + C_{\theta_\eta} \theta_\eta + C_{\theta'_\eta} \theta'_\eta + C_{\theta_\eta \theta'_\eta} \theta_\eta \theta'_\eta, \quad (5.58)$$

where we have

$$C = -X_- \chi_\eta - \eta \cdot \zeta \frac{D_5}{2P_+} A + \frac{\kappa_5}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) \chi_\eta - \frac{\kappa_5}{2P_+} (\eta \cdot \zeta)^2 \chi'_\eta, \quad (5.59)$$

$$C_{\theta_\eta} = -\frac{P^2 - \mathcal{M}^2}{2P_+} A + \frac{B}{P_+} - X_- B + \eta \cdot \zeta \frac{D_5}{2P_+} \chi'_\eta + \frac{\kappa_5}{2P_+} (\eta \cdot \zeta)(\eta' \cdot \zeta) B, \quad (5.60)$$

$$C_{\theta'_\eta} = -\eta \cdot \zeta \frac{D_5}{2P_+} \chi_\eta - \kappa_5 (\eta \cdot \zeta)^2 \frac{B}{2P_+}, \quad (5.61)$$

$$C_{\theta_\eta \theta'_\eta} = \frac{P^2 - \mathcal{M}^2}{2P_+} \chi_\eta - \eta \cdot \zeta \frac{D_5}{2P_+} B. \quad (5.62)$$

Notice the apparent similarity between equations (5.61) and (5.62), these can in fact be shown to be a linear transformation of each other. Firstly, note that

$$D_5^2 = \gamma_5(\gamma \cdot P - \mathcal{M})\gamma_5(\gamma \cdot P - \mathcal{M}) = -\kappa_5(P^2 - \mathcal{M}^2), \quad (5.63)$$

thus we can write equation (5.62)

$$\begin{aligned} C_{\theta_\eta \theta'_\eta} &= -\frac{D_5^2}{2\kappa_5 P_+} \chi_\eta - \frac{(\eta \cdot \zeta) D_5}{2P_+} B, \\ &= -\frac{D_5}{\kappa_5 (\eta \cdot \zeta)} \left(\frac{(\eta \cdot \zeta) D_5}{2P_+} \chi_\eta + \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} B \right). \end{aligned} \quad (5.64)$$

Thus it can be seen that equations (5.61) and (5.62) differ only by a factor of $-D_5/(\kappa_5(\eta \cdot \zeta))$. Note that if

$$\frac{(\eta \cdot \zeta) D_5}{2P_+} \chi_\eta + \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} B = 0,$$

then both equations (5.61) and (5.62) will be zero. It is interesting to note that a similar situation exists in equations (5.59) and (5.60). The common component of these two equations is

$$\frac{D_5}{2P_+}A + \kappa_5 \frac{(\eta \cdot \zeta)}{2P_+} \chi'_\eta.$$

Taking these similarities between the two pairs of equations into account we can redefine the expansion of ψ as follows: rescale χ_η and A by

$$\chi_\eta \equiv \tilde{\chi}_\eta + (\eta \cdot \zeta) \frac{D_5}{P^2 - \mathcal{M}^2} B, \quad (5.65)$$

$$A \equiv \tilde{A} + (\eta \cdot \zeta) \frac{D_5}{P^2 - \mathcal{M}^2} \chi'_\eta, \quad (5.66)$$

and so ψ becomes

$$\psi = \left(\tilde{A} + (\eta \cdot \zeta) \frac{D_5}{P^2 - \mathcal{M}^2} \chi'_\eta \right) + \theta_\eta \chi'_\eta - \theta'_\eta \left(\tilde{\chi}_\eta + (\eta \cdot \zeta) \frac{D_5}{P^2 - \mathcal{M}^2} B \right) + \theta_\eta \theta'_\eta B. \quad (5.67)$$

Using this redefinition equation (5.61) becomes

$$\begin{aligned} C_{\theta'_\eta} &= -\eta \cdot \zeta \frac{D_5}{2P_+} \tilde{\chi}_\eta - \frac{(\eta \cdot \zeta)^2}{2P_+} \frac{D_5^2}{P^2 - \mathcal{M}^2} B - \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} B, \\ &= -\eta \cdot \zeta \frac{D_5}{2P_+} \tilde{\chi}_\eta + \frac{(\eta \cdot \zeta)^2}{2P_+} \frac{\kappa_5 (P^2 - \mathcal{M}^2)}{P^2 - \mathcal{M}^2} B - \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} B, \\ &= -\eta \cdot \zeta \frac{D_5}{2P_+} \tilde{\chi}_\eta. \end{aligned} \quad (5.68)$$

By enforcing $\Omega\psi = 0$ we get $C_{\theta'_\eta} = 0$, which by (5.68) gives the Dirac equation

$$-\eta \cdot \zeta \frac{D_5}{2P_+} \tilde{\chi}_\eta = 0. \quad (5.69)$$

Similarly we can rewrite equation (5.62) as

$$C_{\theta_\eta \theta'_\eta} = \frac{P^2 - \mathcal{M}^2}{2P_+} \tilde{\chi}_\eta = \frac{1}{\kappa_5} D_5 (D_5 \tilde{\chi}_\eta), \quad (5.70)$$

which, by enforcing $\Omega\psi = 0$ leads to the Klein-Gordon equation

$$D_5^2 \tilde{\chi}_\eta = 0. \quad (5.71)$$

Under the rescaling of equation (5.66) equation (5.59) becomes

$$\begin{aligned} C &= \left(-X_- + \frac{\kappa_5}{2P_+} (\eta \cdot \zeta) (\eta' \cdot \zeta) \right) \chi_\eta - (\eta \cdot \zeta) \frac{D_5}{2P_+} \tilde{A} + \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} \chi'_\eta \\ &\quad - \kappa_5 \frac{(\eta \cdot \zeta)^2}{2P_+} \chi'_\eta, \end{aligned} \quad (5.72)$$

and by substituting equation (5.49) into (5.72) we get

$$C = \left(-X_- + \frac{4\ell - \kappa_5 + 4}{4P_+} \right) \chi_\eta - (\eta \cdot \zeta) \frac{D_5}{2P_+} \tilde{A} = 0. \quad (5.73)$$

Similarly, using equations (5.66) and (5.50), (5.60) can now be written

$$C_{\theta_\eta} = \frac{D_5^2}{2\kappa_5 P_+} \tilde{A} - \left(X_- - \frac{4\ell - \kappa_5 + 4}{4P_+} \right) B = 0, \quad (5.74)$$

where once again we have enforced the condition for physical states.

Defining the symbol

$$\tilde{X}_- = X_- - \frac{4\ell - \kappa_5 + 4}{4P_+}, \quad (5.75)$$

and multiplying equation (5.74) by $(\eta \cdot \zeta) \frac{D_5}{P_-^2 - \mathcal{M}^2}$ and then subtracting it from equation (5.73) we get

$$-\tilde{X}_- \left(\chi_\eta - (\eta \cdot \zeta) \frac{D_5}{P_-^2 - \mathcal{M}^2} B \right) = 0. \quad (5.76)$$

Substituting equation (5.65) into (5.76) gives the third equation of motion

$$-\tilde{X}_- \tilde{\chi}_\eta = 0. \quad (5.77)$$

Finally, we identify the physical states at generic ghost number ℓ , arising from the cohomology of Ω , as the spinors $\tilde{\chi}$, with the following properties. From (5.69), the $\tilde{\chi}$ obey the usual massive Dirac equation. From (5.18), (5.56), the P_- constraint dictates that the dependence of superfield components on light cone time $\tau = x^-$, is $P_- = \partial/\partial x^-$. Finally, interpreting (5.76), (5.77) in the p_+ -representation (the Fourier transform of the x^+ -representation, *i.e.* $X_- \equiv X^+ = -\partial/\partial p_+$), the $\tilde{\chi}$ are homogeneous functions of p_+ of degree $-(+1 - \frac{1}{4}\kappa_5 + \ell)$. Thus the $\tilde{\chi}$ are essentially only functions over $(d-1)+1$ -dimensional Minkowski space. For example, in the case $\kappa_5 = 1, \ell = -3/4$ (which implies $\Lambda = 0$), and using (A.16), we find that the physical states have the following explicit form[†]

$$|\tilde{\chi}_\eta\rangle = \tilde{\chi}_\eta(x^\mu) [|0, 0\rangle - |1, 1\rangle + |2, 2\rangle + \dots + (-1)^m |m, m\rangle + \dots], \quad (5.78)$$

[†] $-2\ell = 3/2$ is the correct conformal dimension for a spinor field (see [53])

in terms of the number states of the Bosonic ghost sector (see appendix A), where $\tilde{\chi}_\eta(x^\mu)$ are ordinary functions of x^μ .

This analysis has unearthed a range of allowable ghost numbers, all of which correspond to physical states arrived at through Dirac quantisation. As such there is no maximal ghost number, and so we can't rely on the triviality of $Im\Omega$ to ensure the uniqueness a particular solution. Consequently for any given solution $\tilde{\chi}$ there is a family of corresponding solutions defined through equation (5.68) which take the form $\tilde{\chi} - \eta \cdot \zeta \frac{D_5}{2P_+} \tilde{\chi}_\eta$, for some arbitrary spinor $\hat{\chi}_\eta$, and with the tilde function defined using (5.65).

5.5 BFV-BRST quantisation of the spinning particle and $iosp(d, 2/2)$ structure

As is well known [14, 17], the BFV canonical quantisation of constrained Hamiltonian systems [5, 6, 7] uses an extended phase space description in which, to each first class constraint ϕ_a , a pair of conjugate ‘ghost’ variables (of Grassmann parity opposite to that of the constraint) is introduced. Here we follow this procedure for the spinning relativistic particle. Although our notation is adapted to the massive case, $\mathcal{M} > 0$, as would follow from the second-order action corresponding to extremisation of the proper length of the particle world-line, an analysis of the *fundamental* Hamiltonian description of the first-order action[14] leads to an equivalent picture (with an additional mass parameter $\mu \neq 0$ supplanting m in appropriate equations, and permitting $m \rightarrow 0$ as a smooth limit).

In either case, for the scalar or spinning particle the primary first class constraint is the mass-shell condition $\phi_1 = (P^2 - \mathcal{M}^2)$, where $P^2 = P^\mu \eta_{\mu\nu} P^\nu$. Including the Lagrange multiplier λ as an additional dynamical variable leads to a second constraint, reflecting conservation of its conjugate momentum π_λ . The quantum formulation should be consistent with the equations of motion and gauge fixing at the classical level, as two such restrictions are necessary so as to arrive at the particle quantisation corresponding with the superalgebraic prescription

of Section 5.3. Firstly, we choose below to work in the class [26, 37, 38, 9] $\dot{\lambda} = 0$; moreover, we take gauge fixing only to be with respect to gauge transformations in the *connected* component of the group, *i.e.* the identity class, or the disconnected component, *i.e.* the orientation reversing class characterised by $\tau' = \tau_i + \tau_f - \tau$. Thus λ will be quantised on the half line (say \mathbb{R}^+), and the system is not modular-invariant until the two distinctly oriented sectors (particle and anti-particle) are combined [14]. Secondly, we take $\phi_1^{(2)} = \lambda\pi_\lambda$ as the other secondary first class constraint (rather than $\phi_1^{(2)} = \pi_\lambda$ used in the standard construction). Finally, the spinning particle system also entails a second, Grassman odd first class constraint $\phi_2 = p_\mu\zeta^\mu + \mathcal{M}\epsilon\gamma_5$ ($\epsilon = \pm 1$), together with its associated first class constraint $\phi_2^{(2)} = \pi_2^{(2)}$, the conjugate momentum of the corresponding Lagrange multiplier λ_2 (which is also Grassmann odd).

5.5.1 BFV extended state space and wavefunctions

The BFV extended phase space [14] for the BRST quantisation of the spinning relativistic particle is therefore taken to comprise the following canonical variables:

$$x^\mu(\tau), p_\mu(\tau), \zeta^\mu(\tau), \zeta_5(\tau), \lambda(\tau), \pi_\lambda(\tau), \lambda_2(\tau), \pi_2(\tau), \eta^{a(i)}(\tau), \rho_{a(i)}(\tau), \quad a, i = 1, 2. \quad (5.79)$$

$x^\mu(\tau), p_\mu(\tau)$ are Grassmann even whilst ζ^μ, ζ_5 are Grassmann odd variables, λ is the Grassmann even Lagrange multiplier corresponding to the even first class constraint ϕ_1 , π_λ is the momentum conjugate to λ (which forms the constraint $\phi_1^{(2)}$), λ_2 is the odd Lagrange multiplier corresponding to the Grassmann odd first class constraint ϕ_2 , and $\phi_2^{(2)} = \pi_2$ is its conjugate momentum. $\eta^{1(1)}, \rho_{1(1)}$ and $\eta^{1(2)}, \rho_{1(2)}$ are the Grassmann odd conjugate pairs of ghosts corresponding to the constraints ϕ_1 and $\phi_1^{(2)}$ respectively, while $\eta^{2(1)}, \rho_{2(1)}$ and $\eta^{2(2)}, \rho_{2(2)}$ are the Grassmann even conjugate pairs of ghosts corresponding to the constraints ϕ_2 and $\phi_2^{(2)}$ respectively. We proceed directly to the quantised version by introducing the Schrödinger representation. We introduce operators X^μ, P_ν corresponding

to the coordinates x^μ, p_ν , acting on suitable sets of wavefunctions over x^μ , and on the half line $\lambda > 0$. The Hermitian ghosts $\eta^{a(i)}, \rho_{b(j)}$ (a pair of bc systems) are represented as usual either on a 4-dimensional indefinite inner product space $|\sigma\sigma'\rangle$, $\sigma, \sigma' = \pm$, or here, in order to match with Section 5.3, in terms of suitable Grassmann variables acting on superfields. The non zero commutation relations amongst (5.79) read (repeated in full for clarity):

$$\begin{aligned} [X_\mu, P_\nu] &= -i\eta_{\mu\nu}, \quad \{\zeta_\mu, \zeta_\nu\} = -\frac{2}{\kappa_5}\eta_{\mu\nu}, \quad \{\gamma_5, \gamma_5\} = 2\kappa_5, \\ [\lambda, \pi_\lambda] &= i, \quad \{\lambda_2, \pi_2\} = -i, \\ \{\eta^{1(i)}, \rho_{1(j)}\} &= -i\delta_j^i, \quad [\eta^{2(i)}, \rho_{2(j)}] = i\delta_j^i \quad i, j = 1, 2, \end{aligned} \quad (5.80)$$

from which the algebra of constraints follows:

$$\{\phi_2, \phi_2\} = -2\kappa_5\phi_1 \quad [\phi_1, \phi_2] = [\phi_1, \phi_1] = 0. \quad (5.81)$$

The Hermiticity conditions imposed on the above operators read

$$\begin{aligned} X_\mu^\dagger &= X_\mu, \quad P_\mu^\dagger = P_\mu, \quad \zeta_\nu^\dagger = \zeta_\nu, \quad \mu, \nu = 0, \dots, d-1, \\ \zeta_0^\dagger &= -\zeta_0, \quad \gamma_5^\dagger = \gamma_5, \\ \lambda^\dagger &= \lambda, \quad \pi_\lambda^\dagger = \pi_\lambda, \quad \lambda_2^\dagger = -\lambda_2, \quad \pi_2^\dagger = \pi_2, \\ (\eta^{a(i)})^\dagger &= \eta^{a(i)}, \quad (\rho_{a(i)})^\dagger = (-1)^{a+1}(\rho_{a(i)}), \quad a = 1, 2. \end{aligned} \quad (5.82)$$

The ghost number operator N_{gh} is defined by

$$N_{\text{gh}} = \frac{i}{2} \sum_{a,i=1}^2 (\eta^{a(i)} \rho_{a(i)} - (-1)^{(a-1)} \rho_{a(i)} \eta^{a(i)}). \quad (5.83)$$

The canonical BRST operator[§] is given by

$$\Omega = \eta^{1(1)}\phi_1 + \eta^{1(2)}\phi_1^{(2)} + \eta^{2(1)}\phi_2 + \eta^{2(2)}\phi_2^{(2)} + \frac{i}{2} (\eta^{2(1)})^2 \rho_{1(1)}. \quad (5.84)$$

The admissible gauge fixing operator [26] \mathcal{F} , which is free of Gribov problems and leads to the appropriate effective Hamiltonian is given by:

$$\mathcal{F} = -\frac{1}{2}\lambda\rho_{1(1)}, \quad (5.85)$$

[§]The criteria for the construction and nilpotency of the corresponding anti-BRST operator $\bar{\Omega}$ have been given in [32]

and thus the Hamiltonian can be written

$$H = i \llbracket \mathcal{F}, \Omega \rrbracket = -\frac{1}{2} \lambda \left(\eta^{1(2)} \rho_{1(1)} + \phi_1 \right), \quad (5.86)$$

which is of course BRST-invariant.

Consider the following canonical transformations on the classical dynamical variables of the extended phase space [25]

$$\begin{aligned} \eta'^{a(i)} &= \lambda \eta^{a(i)}, & a(i) &= 1(1), 1(2), 2(1), \\ \rho'_{a(i)} &= \frac{1}{\lambda} \rho_{a(i)}, \end{aligned} \quad (5.87)$$

$$\lambda \pi'_\lambda = \lambda \pi_\lambda + (-\eta^{1(1)} \rho_{1(1)} + \rho_{1(2)} \eta^{1(2)} - \eta^{2(1)} \rho_{2(1)}), \quad (5.88)$$

with the remainder invariant. At the same time we relabel the coordinates $p_+ = \lambda^{-1}$ and $x_- = \lambda \pi_\lambda$. At the quantum level the corresponding BRST operator

$$\Omega' = \eta'^{1(1)} \phi_1 + \eta'^{1(2)} \phi'_1{}^{(2)} + \eta'^{2(1)} \phi_2 + \eta'^{2(2)} \phi_2^{(2)} + \frac{1}{2} \left(\eta'^{2(1)} \right)^2 \rho'_{1(1)},$$

can be written as

$$\begin{aligned} \Omega' &= \lambda \eta^{1(1)} \phi_1 + \eta^{1(2)} : \lambda \phi_1^{(2)} : + \lambda \eta^{2(1)} \phi_2 + \eta^{2(2)} \phi_2^{(2)} \\ &\quad - \lambda \eta^{1(2)} \eta^{1(1)} \rho_{1(1)} - \lambda \eta^{1(2)} \eta^{2(1)} \rho_{2(1)} + \frac{i}{2} \lambda \left(\eta^{2(1)} \right)^2 \rho_{1(1)}, \end{aligned} \quad (5.89)$$

where the symmetric ordering

$$: \lambda \phi_1^{(2)} : := \frac{1}{2} (\lambda \phi_1^{(2)} + \phi_1^{(2)} \lambda) = \lambda \phi_1^{(2)} - \frac{1}{2} \lambda,$$

has been introduced.

It is also convenient to define [25] the operators $\theta_\alpha, Q_\alpha, \zeta'_\alpha$ and $\tilde{\zeta}'_\alpha$ ($\alpha = 1, 2$) by

$$\begin{aligned} Q_{1,2} &= \frac{i}{2\sqrt{2}} (2\eta^{1(2)} \pm \rho_{1(1)}), & \theta_{1,2} &= \frac{i}{\sqrt{2}} (\pm \rho_{1(2)} - 2\eta^{1(1)}), \\ \zeta'_{1,2} &= \frac{1}{\sqrt{2}} (\eta^{2(1)} \pm \rho_{2(1)}), & \tilde{\zeta}'_{1,2} &= \frac{1}{\sqrt{2}} (\pm \eta^{2(2)} - \rho_{2(2)}), \end{aligned} \quad (5.90)$$

which obey the relations

$$\{Q_\alpha, \theta_\beta\} = i\varepsilon_{\alpha\beta} \quad \text{and} \quad [\zeta'_\alpha, \zeta'_\beta] = -\varepsilon_{\alpha\beta}. \quad (5.91)$$

In terms of these variables we obtain the following simple forms for the BRST, gauge fixing and Hamiltonian operators.

$$\begin{aligned}
\Omega' &= -\frac{i}{\sqrt{2}} \left(: \lambda \phi_1^{(2)} : (Q_1 + Q_2) + (\theta_1 + \theta_2) H \right. \\
&\quad \left. + i(\zeta'_1 + \zeta'_2) \lambda (\phi_2 + Q^\alpha \zeta'_\alpha) + i(\tilde{\zeta}'_1 - \tilde{\zeta}'_2) \phi_2^{(2)} \right), \\
\mathcal{F}' &= -\frac{1}{2} \rho_{1(1)} = -\frac{1}{\sqrt{2}} (Q_1 - Q_2), \\
H' &= i [\mathcal{F}', \Omega'] = -\frac{\lambda}{2} (P^\mu P_\mu + Q^\alpha Q_\alpha - \mathcal{M}^2) \equiv H.
\end{aligned} \tag{5.92}$$

Note that the ζ'_α defined here and the ζ_α defined in section 5.3.2 differ by a factor $\sqrt{2}$, i.e. $\zeta'_\alpha = \frac{1}{\sqrt{2}} \zeta_\alpha$.

5.5.2 β -limiting procedure for the BRST operator

It is now necessary to reconcile the development of Sections 5.3 and 5.4, in which the identical raw material for construction of the BRST operator, gauge fixing function and hence physical states, appears *purely algebraically* (compare equations (5.30), (5.31), (5.29) with (5.80)) except for the absence of the $\eta^{2(2)}, \rho_{2(2)}$ even ghosts, and thus the $\tilde{\zeta}_1, \tilde{\zeta}_2$ oscillators. In [25], a somewhat heuristic argument was provided to justify the restriction to the vacuum of the latter oscillators. Here instead we shall use what is known as the β -limiting procedure [17, 14] applied throughout on the $a = 2$ label of the BFV phase space variables (if we also apply it to the $a = 1$ label we recover the Fadeev-Popov reduced phase space quantisation scheme). The exposition will closely follow that of [14].

Consider instead of (5.85) the gauge fixing Fermion

$$\mathcal{F} = -\frac{1}{2} \lambda \rho_{1(1)} + \frac{1}{\beta} (\lambda_2 - \lambda_2^0) \rho_{2(2)} + \lambda_2 \rho_{2(1)}, \tag{5.93}$$

where β is arbitrary, real and Grassmann even, and λ_2^0 is some given function of time with the same properties and Grassmann parities as λ_2 . The Hamiltonian is thus given by

$$\begin{aligned}
H_{eff} = i [\mathcal{F}, \Omega] &= -\frac{1}{2} \lambda (\phi_1 + \eta^{1(2)} \rho_{1(1)}) + \frac{1}{\beta} (\lambda_2 - \lambda_2^0) \phi_2^{(2)} + \frac{1}{\beta} \eta^{2(2)} \rho_{2(2)} \\
&\quad + \eta^{2(2)} \rho_{2(1)} + \lambda_2 \eta^{2(1)} \rho_{1(1)} + \lambda_2 \phi_2.
\end{aligned} \tag{5.94}$$

The equations of motion for the BFV phase space variables can be easily obtained for the above H by implementing as usual $\dot{A} = i \llbracket A, H_{eff} \rrbracket$. Those which are different from that calculated using (5.86) can be written

$$\begin{aligned}\dot{\lambda}_2 &= -\frac{i}{\beta}(\lambda_2 - \lambda_2^0), \\ \dot{\pi}_2 &= -i \left(\frac{1}{\beta} \pi_2 + \eta^{2(1)} \rho_{1(1)} + \phi_2 \right), \\ \dot{\eta}^{2(2)} &= \frac{1}{\beta} \eta^{2(2)}, \\ \dot{\rho}_{2(2)} &= \frac{1}{\beta} \rho_{2(2)} + \rho_{2(1)}.\end{aligned}$$

If we now change to new variables $\tilde{\pi}_2, \tilde{\rho}_{2(2)}$ such that

$$\pi_2 = \beta \tilde{\pi}_2, \quad \rho_{2(2)} = \beta \tilde{\rho}_{2(2)}$$

and substitute these into H_{eff}, Ω and N_{gh} , the equations of motion and the action related to H_{eff} . *Only* after having done that we take the limit $\beta \rightarrow 0$. This leads to the BRST and ghost number operators becoming

$$\Omega = \eta^{1(1)} \phi_1 + \eta^{1(2)} \phi_1^{(2)} + \eta^{2(1)} \phi_2 + \frac{i}{2} (\eta^{2(1)})^2 \rho_{1(1)}, \quad (5.95)$$

$$N_{gh} = i \sum_{i=1}^2 \eta^{1(i)} \rho_{1(i)} + \eta^{2(1)} \rho_{2(2)} - \frac{1}{2} \quad (5.96)$$

while the equations of motion for $\lambda_2, \pi_2, \eta^{2(2)}$ and $\rho_{2(2)}$ are

$$(\lambda_2 - \lambda_2^0) = 0, \quad \tilde{\pi}_2 = -\phi_2 - \eta^{2(1)} \rho_{1(1)}, \quad \eta^{2(2)} = 0, \quad \rho_{2(2)} = -\tilde{\rho}_{2(2)}. \quad (5.97)$$

Solving these equations and taking $\lambda_2^0 = 0$, we find that equations (5.95) and (5.96) remain as they are whilst $H_{eff} = H$. Note in particular that through the β -limiting procedure, the function Ψ in 5.93 leads to the gauge fixing conditions $\lambda_2 = 0$, with $\tilde{\pi}_2$ being the associated Lagrange multipliers. Thus we have succeeded in ‘squeezing out’ the 2(2) pair of even ghosts together with the odd Lagrange multiplier λ_2 . Moreover the Hamiltonian in equation (5.86), obtained from the admissible gauge fixing Fermion given in equation (5.85), is recovered.

Finally, and most importantly, the canonical transformation in equations (5.87) and (5.88) is not affected by this procedure, as can easily be observed. Thus

whether we apply the canonical transformations before the β -limiting procedure or after does not matter. Consequently via equations (5.87), (5.88) equation (5.95) becomes

$$\Omega' = -\frac{i}{\sqrt{2}} \left(: \lambda \phi_1^{(2)} : (Q_1 + Q_2) + (\theta_1 + \theta_2)H + i(\zeta'_1 + \zeta'_2)\lambda(\phi_2 + Q^\alpha \zeta_\alpha) \right). \quad (5.98)$$

The forms (5.86) and (5.98) can now be shown to be identical to the previously given algebraically defined expressions for these quantities (5.18), (5.36). The raw material (5.30), (5.31), (5.29) also appears in this construction, as can be easily observed by (5.80), and by identifying $P_+ = \lambda^{-1}$, $X_- =: \lambda \phi_1^{(2)} :$, $\zeta_- = \phi_2 + Q^\alpha \zeta'_\alpha$ and the BRST operator $\Omega' = \eta^\alpha L_{\alpha-}$. Moreover, the realisation of $iosp(d, 2/2)$ can be done as in (5.20), (5.25). In particular, the evaluation of the BRST cohomology performed in Section 5.4 above, gives precisely the correct identification of physical state wavefunctions for the spinning particle model of this section, provided that we represent ζ'_α by $(-1)^z \zeta'_\alpha$ to account for the correct action on the superfield and the correct commutation relations of $iosp(d, 2/2)$. The constant κ_5 appearing in (5.18) can also be introduced in the third equation of (5.92) to account for $\gamma_5 = \pm 1$, which will eventually appear in the factorisation of P_- leading to the Dirac equation.

5.6 Conclusions

This chapter, via the positive results claimed here for the case of the spinning particle, provides further confirmation of our program of establishing the roots of covariant quantisation of relativistic particle systems, in the BRST complex associated with representations of classes of extended spacetime supersymmetries (in this case $iosp(d, 2/2)$ symmetry). The results contained in this chapter have been published in a similar form in reference [33].

The approach taken in this chapter is similar to that in chapter 4, except here the method was expanded upon and implemented in a system with Fermionic degrees of freedom. This involved the introduction of an extended Clifford algebra

with generators Γ_N , containing both Bosonic and Fermionic coordinates. In the Clifford algebra the mass-shell condition can be factorized, allowing the Dirac condition to be imposed. This allowed us to split the Dirac wavefunctions so that the $iosp(d, 2/2)$ algebra was effectively realised on $2^{d/2}$ -dimensional Dirac spinors. When applying the standard Hamiltonian BFV-BRST *Ansatz* it was necessary to introduce the concept of β -limiting so as identify the appropriate sector of the full phase space.

The spinning particle is one of the simplest examples of a supersymmetric system, and thus the success of this chapter provides a firm foundation upon which further study of supersymmetric systems can be based. Such future work will be discussed further in chapter 8

Chapter 6

The $D(2, 1; \alpha)$ particle

In the previous two chapters we examined the covariant BFV-BRST quantisation of the scalar and spinning particle respectively. In both these chapters we started with a physical model for the system, which was then quantised and shown to obey the $iosp(d, 2/2)$ algebra. In this chapter we take an algebraic approach; as $osp(d, 2/2)$ is a member of the class of classical simple Lie superalgebras, by an appropriate generalisation it should be possible to extend the quantisation superalgebra $iosp(d, 2/2)$ into a more general classical simple Lie superalgebra. The motivation behind this is the need for a characterisation of admissible spacetime ‘BFV-BRST extended’ supersymmetries in various dimensions. In this chapter we demonstrate this by studying the particular case of $d = 2$, which leads to the quantisation of two-dimensional relativistic particles in the exceptional superalgebra $D(2, 1; \alpha)$.

In section 6.1 we briefly define and review the properties of the exceptional superalgebra $D(2, 1; \alpha)$. In section 6.2 we shall construct superfield representations of the BFV-BRST quantisation superalgebra corresponding to $D(2, 1; \alpha)$ and study the physical states via the BRST operator. This will be done using only the algebraic structure as a guide (*i.e.* no physical model). Finally, in section 6.3 we shall reverse-engineer a classical action corresponding to the algebraic model we have constructed, and identify the corresponding Lagrangian equations of motion. A preliminary version of the results contained in this chapter can be

seen in [54], whilst the full results will be published in a paper in preparation [55].

6.1 The exceptional superalgebra $D(2, 1; \alpha)$

The classical simple Lie superalgebras consist of the $spl(m/n)$ and the $osp(m/2n)$ families, the strange series $P(n)$ and $Q(n)$ and the exceptional algebras $F(4)$, $G(3)$ and $D(2, 1; \alpha)$. Comprehensive definitions and descriptions of these algebras can be found in several places, see for example [56, 57, 58, 59]. The orthosymplectic families, of which $osp(d, 2/2)$ is a member, along with the general linear superalgebras have been described in [60, 61, 62]. The exceptional superalgebras $F(4)$ and $G(3)$ have been studied in [63]. Finally, a study of the $D(2, 1; \alpha)$ algebras, including a detailed analysis of their finite and infinite-dimensional irreducible representations has been carried out by Van der Jeugt [64]. The explicit supercommutation relations of the $D(2, 1; \alpha)$ superalgebras are given in [65]. If the reader desires a more thorough description of the $D(2, 1; \alpha)$ algebras then they are directed to any of the above mentioned references.

The algebras $D(2, 1; \alpha)$ are a one-parameter family of 17-dimensional non-isomorphic Lie superalgebras. For the special case of $\alpha = 1$ we have $D(2, 1; 1) \cong osp(2, 2/2)$. It is through this special case that we seek to generalise the BFV-BRST quantisation algebra. This aspect will be discussed in more detail in the next section.

The $D(2, 1; \alpha)$ algebras are sometimes denoted $\Gamma(\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1, \sigma_2, \sigma_3$ are three complex variables. This labelling gives the impression of a three parameter family of superalgebras, however due to certain conditions [57] there is effectively only one parameter. See [64] for the relationship between the two forms of labellings of $D(2, 1; \alpha)$.

The even part of the superalgebra $D(2, 1; \alpha)$ is the 9-dimensional non-compact form $sl(2, \mathbb{R}) + sl(2, \mathbb{R}) + sl(2, \mathbb{R})$, whilst the odd part (of dimension 8) is the spinorial representation $(2, 2, 2)$ of the even part. The parameter α appears only

in the anti-commutation relations among the components of the tensor products (*i.e.* the odd components). In terms of the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ such that $\varepsilon_1^2 = -(1 + \alpha)/2$, $\varepsilon_2^2 = 1/2$, $\varepsilon_3^2 = \alpha/2$ and $\varepsilon_i \cdot \varepsilon_j = 0$ if $i \neq j$, the root system $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ is given by

$$\Delta_{\bar{0}} = \{\pm 2\varepsilon_i\} \quad \text{and} \quad \Delta_{\bar{1}} = \{\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}.$$

In [66], Günaydin studies $D(2, 1; \alpha)$ considered as the superconformal symmetry group of an exotic family of superspaces in two dimensions defined by the one-parameter family of Jordan superalgebras $JD(2/2)_\alpha$. In this paper he also derives the full super-differential operators representing the actions of $D(2, 1; \alpha)$ on the exotic superspaces. In this chapter we also derive a superfield realisation, however it is over a different superspace. Finally, by treating $D(2, 1; \alpha)$ as a superconformal group, Günaydin also shows a relationship between this chapter and the future work on the unification of BRST and conformal symmetries in BFV quantisation mentioned in chapter 8.

6.2 The quantised $D(2, 1; \alpha)$ particle

The BFV-BRST quantisation of relativistic systems provides a cohomological resolution of irreducible unitary representations (unirreps) of space-time symmetries. Moreover these unirreps appear to be associated with constructions of $iosp(d, 2/2)$ for relativistic particles in flat spacetime, as can be seen in chapters 4 and 5. In this chapter, however, we do involve translations as additional generators and so the algebra reduces to $osp(d, 2/2)$. Here we follow an algebraic approach, and so need to develop a classification of admissible ‘quantisation superalgebras’ in various dimensions. Some examples of such algebras [66, 67] are $D(2, 1; \alpha)$ in $d = 1 + 1$ (note that as $\alpha = 1$ corresponds to $osp(2, 2/2)$), in $D = 2 + 1$ we have $osp(3, 2/2)$ which corresponds to anti de Sitter symmetry (which may thus be relevant to anyon quantisation), and for $d = 3 + 1$ we get conformal symmetry of $4D$ spacetime and super unitary superalgebras as possible alternative quantisation superalgebras.

In order to detail our construction, it is necessary to represent the scalar particle (in $d = 2$) in a manner more along the lines of Govaerts in [14]. In this form the second-order action for the scalar particle can be written

$$S = m \int_{\tau_i}^{\tau_f} d\tau \sqrt{\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu}}, \quad (6.1)$$

which leads to the canonical momenta

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m \frac{\dot{x}_\mu}{\dot{x}^2}.$$

In accordance with the BRST prescription, we enlarge the phase space by treating the Lagrange multiplier λ as a dynamical variable, along with introducing an associated vanishing conjugate momentum π_λ . The system has two constraints; the first being the mass-shell condition

$$\phi_1 = p^\mu p_\mu - m^2 = 0,$$

and the second the aforementioned momenta conjugate to the Lagrange multiplier. The Poisson brackets are the usual ones: $\{p_\mu, x^\nu\} = \delta_\mu^\nu$, $\{\lambda, \pi_\lambda\} = 1$.

The extended action (6.1) is invariant under the following infinitesimal gauge transformations

$$\begin{aligned} \delta \lambda &= \dot{\varepsilon}, \\ \delta x^\mu &= \{x^\mu, \varepsilon \phi_1\} = 2\varepsilon p^\mu, \\ \delta p^\mu &= \{p^\mu, \varepsilon \phi_1\} = 0, \end{aligned}$$

with $\varepsilon(\tau)$ being an arbitrary (dimensionless) infinitesimal function such that $\varepsilon(\tau_i) = \varepsilon(\tau_f) = 0$.

In order to derive the equations of motion for the action (6.1) it is necessary to choose a particular gauge fixing condition. From the transformations of the einbein (which arises explicitly in the first-order formulation, as can be seen in chapter 4) under world-line diffeomorphisms, we have

$$\frac{d\tilde{\tau}}{d\tau} = \frac{e(\tau)}{\tilde{e}(\tilde{\tau})}.$$

It is thus always possible to find a parametrisation $\tilde{\tau}(\tau)$ such that $\tilde{e}(\tilde{\tau})$ is a constant. In the infinitesimal case we can write

$$\tau = \tilde{\tau} + \frac{\varepsilon(\tilde{\tau})}{e(\tau)},$$

and so we have

$$\delta e = \tilde{e}(\tau) - e(\tau) = \dot{\varepsilon}.$$

Using this relationship allows us to identify the Lagrange multiplier $\lambda(\tau)$ with the einbein $e(\tau)$.

As in the two previous chapters (see sections 4.5 or 5.5) it is necessary to place two restrictions on the system so as to arrive at the particle quantisation corresponding to the superalgebraic prescription. Firstly, we take gauge fixing with respect to gauge transformations in either the connected or disconnected component of the group. This is equivalent to limiting the quantisation of λ to the half line (\mathbb{R}^+ or \mathbb{R}^-). Secondly we take as a first class constraint $\phi_2 = \lambda\pi_\lambda$ (rather than the usual $\phi_2 = \pi_\lambda$ used in the standard construction).

If we now extend the phase space and in the same manner to that given in section 4.5.1 or [8] then we can write the standard BRST operator

$$\Omega = \eta^1 \phi_1 + \eta^2 \phi_2.$$

With a gauge fixing function defined as $\mathcal{F} = -\frac{1}{2}\lambda\rho_2$ the Hamiltonian can be calculated as

$$H = \{\mathcal{F}, \Omega\} = -\frac{1}{2}\lambda (\eta^1 \rho_2 + p^\mu p_\mu - m^2). \quad (6.2)$$

Similarly, quantising the system via the standard Schrödinger representation we have the operators

$$X^\mu, P_\nu, \lambda, \pi_\lambda, Q_\alpha, X_\beta, P_+, X_-, P_- = H, X_+ = \tau. \quad (6.3)$$

The non-zero commutation relations between these operators can be seen in equation 4.27, or in [8] (except that we have written $X_\alpha = \theta_\alpha$). Upon doing this we find that the operators J_{AB} defined as in (4.28) generate the inhomogeneous orthosymplectic superalgebra $isop(D, 2/2)$, whose commutation relations are given in (4.17) and (4.18).

For simplicity, we recognise that $J_{\mu\nu}$ is anti-symmetric. This allows us to define the operator J by

$$J_{\mu\nu} = \varepsilon_{\mu\nu} J. \quad (6.4)$$

In order to change $osp(2, 2/2)$ to $D(2, 1; \alpha)$, we must modify three of the anti-commutation relations given in (4.17) (with the rest remaining the same). The new relations are

$$\begin{aligned} \{L_{\mu\alpha}, L_{\nu\beta}\} &= \varepsilon_{\alpha\beta} \varepsilon_{\mu\nu} (J + AJ_{+-}) - \eta_{\mu\nu} K_{\alpha\beta} \\ \{L_{\mu\alpha}, L_{\beta\pm}\} &= -\varepsilon_{\alpha\beta} (J_{\mu\pm} \pm B_{\pm} \varepsilon_{\mu}^{\nu} J_{\nu\pm}), \\ \{L_{\alpha\pm}, L_{\beta\mp}\} &= \pm \varepsilon_{\alpha\beta} (J_{+-} \pm C_{\pm} J) - K_{\alpha\beta}. \end{aligned} \quad (6.5)$$

Taking the super-Jacobi identity on $L_{\mu\alpha}, L_{\nu\beta}$ and $L_{\gamma\pm}$ it is straightforward to show that

$$B_{\pm} = \frac{\mp A}{\det(\eta)}.$$

Taking the super-Jacobi identity on $L_{\mu\alpha}, L_{\mu\pm}$ and $L_{\gamma\mp}$ we can show

$$C_{\pm} = \frac{-A}{\det(\eta)}.$$

Through the use of Cartan generators and weight diagrams we can [68] eliminate all but one of A, B_{\pm}, C_{\pm} and relate them back to the α parameter in $D(2, 1; \alpha)$. This results in new generators

$$\begin{aligned} \tilde{J} &= J + aJ_{+-}, \\ \tilde{J}_{+-} &= J_{+-} + aJ, \end{aligned} \quad (6.6)$$

with the single parameter a , given by

$$a = \frac{1 - \alpha}{1 + \alpha}. \quad (6.7)$$

The altered anti-commutation relations (6.5) can now be written

$$\begin{aligned} \{L_{\mu\alpha}, L_{\nu\beta}\} &= \varepsilon_{\alpha\beta} \varepsilon_{\mu\nu} \tilde{J} - \eta_{\mu\nu} K_{\alpha\beta} \\ \{L_{\mu\alpha}, L_{\beta\pm}\} &= -\varepsilon_{\alpha\beta} (J_{\mu\pm} \pm a \varepsilon_{\mu}^{\nu} J_{\nu\pm}), \\ \{L_{\alpha\pm}, L_{\beta\mp}\} &= \pm \varepsilon_{\alpha\beta} \tilde{J}_{+-} - K_{\alpha\beta}. \end{aligned} \quad (6.8)$$

Note that the odd generators close compactly on $\tilde{J}, \tilde{J}_{+-}$ (although at the expense of more complicated commutation brackets). The invariant bilinear form on $D(2, 1; \alpha)$ [68] is

$$\begin{aligned}
(J, J) &= 1, \\
(J, J_{+-}) = (J_{+-}, J) &= -a, \\
(J_{+-}, J_{+-}) &= 1, \\
(J_{\mu\pm}, J_{\nu\pm}) &= -\eta_{\mu\nu} \pm a\varepsilon_{\mu\nu}, \\
(K_{\alpha\beta}, K_{\gamma\delta}) &= (1 - a^2) (\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}), \\
(L_{\mu\alpha}, L_{\nu\beta}) &= (1 - a^2)\eta_{\mu\nu}\varepsilon_{\alpha\beta}, \\
(L_{\alpha\pm}, L_{\beta\mp}) &= (1 - a^2)\varepsilon_{\alpha\beta}.
\end{aligned}$$

And the Casimir C can be written

$$C = C_1 + C_2, \tag{6.9}$$

where,

$$\begin{aligned}
C_1 &= -J^2 - J_{+-}^2 + \{J_+^\mu, J_{\mu-}\} - \frac{1}{2}K^{\alpha\beta}K_{\alpha\beta} - L^{\mu\alpha}L_{\mu\alpha} - [L_+^\alpha, L_{\alpha-}], \\
C_2 &= -a\{J, J_{+-}\} - a\varepsilon^{\mu\nu}\{J_{\mu+}, J_{\nu-}\}.
\end{aligned}$$

For the $D(2, 1; \alpha)$ particle we do not have a physical model such as that laid out between equations (6.1) to (6.3), and so must use the algebraic structure as our only guide. We regard $D(2, 1; \alpha)$ as a generalisation of $osp(d, 2/2)$ and seek to find a superfield realisation which is equivalent to the case for the scalar relativistic particle (as detailed above, in chapter 4 and [8]) for $\alpha = 1$ and $d = 2$.

In the generic d -dimensional $osp(d, 2/2)$ case we can define the homogeneous manifold

$$\mathcal{M} = OSp(d, 2/2)/G_0,$$

where G_0 is the stability group

$$G_0 = OSp(d - 1, 2/2) \wedge \mathcal{N},$$

and

$$\begin{aligned} OSp(d-1, 2/2) &= \langle J_{\mu\nu}, L_{\mu\alpha}, K_{\alpha\beta} \rangle, \\ \mathcal{N} &= \langle J_{\mu-}, L_{\alpha-} \rangle. \end{aligned} \quad (6.10)$$

For one parameter subgroups $g(t)$ with generator A , the standard superfield realisation leads to generators, ϕ acting on \mathcal{M} , defined by

$$\hat{A}\phi(x) = \left(\frac{d}{dt} \Phi(g(t)^{-1}x) \right)_{t=0}, \quad (6.11)$$

where $x \in \mathcal{M}$,

$$x = (q^\mu, \eta^\alpha, \phi) \leftrightarrow \exp(q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}) \exp(\phi J_{+-}) G_0,$$

represents the coset.

In a similar fashion, for $D(2, 1; \alpha)$ we define the homogeneous manifold and stability group as

$$\begin{aligned} \mathcal{M} &= D(2, 1; \alpha) / \tilde{G}_0, \\ \tilde{G}_0 &= OSp(1, 1/2) \wedge \mathcal{N}, \end{aligned}$$

where now $OSp(1, 1/2) = \langle \tilde{J}, L_{\mu\alpha}, K_{\alpha\beta} \rangle$ and \mathcal{N} is unchanged from (6.10). This leads to generators \hat{A} defined as in (6.11), except now the coset is $x \in \mathcal{M}$

$$x \equiv (q^\mu, \eta^\alpha, \phi) \leftrightarrow \exp(q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}) \exp(\phi \tilde{J}_{+-}) \tilde{G}_0.$$

The superfield realisation for $D(2, 1; \alpha)$ can now be calculated by introducing formal group elements $g = e^{\phi \cdot F}$ for the generators F_1, F_2, \dots, F_N and graded parameters $\phi_1, \phi_2, \dots, \phi_N$ and evaluate the product

$$h \cdot g = e^{\epsilon \cdot F} e^{\phi \cdot F},$$

in order to find the product map $\mu(\epsilon, \phi)$ to first-order in ϵ .

Thus we can calculate $J_{\mu+}$ as follows:

Let $\epsilon = e^{\epsilon^\mu J_{\mu+}}$, then

$$\begin{aligned} \epsilon^{-1} \cdot x &= e^{-\epsilon^\mu J_{\mu+}} e^{q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}} e^{\phi \tilde{J}_{+-}} G_0, \\ &= e^{(q^\mu - \epsilon^\mu) J_{\mu+} + \eta^\alpha L_{\alpha+}} e^{\phi \tilde{J}_{+-}} G_0. \end{aligned}$$

So if we consider functions $f(q^\mu, \eta^\alpha, \phi)$, then

$$\epsilon f(q^\mu, \eta^\alpha, \phi) = f(q - \epsilon, \eta, \phi), \quad (6.12)$$

and so

$$\delta f = \epsilon f - f = -\epsilon^\mu \frac{\partial}{\partial q^\mu} f(q^\mu, \eta^\alpha, \phi) + \dots$$

Hence from (6.11) we can write

$$J_{\mu+} = -\frac{\partial}{\partial q^\mu}. \quad (6.13)$$

Finally, for later convenience we re-scale variables as follows $p^\mu = \lambda^{-1} q^\mu$, $\theta^\alpha = \lambda^{-1} \eta^\alpha$ and $\lambda = e^\phi$, ($\lambda > 0$), then we get

$$J_{\mu+} = -\lambda^{-1} \frac{\partial}{\partial p^\mu}.$$

We shall explicitly calculate a further two generators, with the understanding that the remainder can be derived in a similar fashion.

Let $\xi = e^{\xi^\alpha L_{\alpha+}}$, thus

$$\begin{aligned} \xi^{-1} x &= e^{-\xi^\alpha L_{\alpha+}} e^{q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}} e^{\phi \tilde{J}_{+-}} G_0, \\ &= e^{q^\mu J_{\mu+}} e^{\eta^\alpha L_{\alpha+} - \xi^\alpha L_{\alpha+}} e^{\phi \tilde{J}_{+-}} G_0. \end{aligned}$$

Therefore we can see that

$$\delta f = -\xi^\alpha \frac{\partial}{\partial \eta^\alpha} f + \dots,$$

and so we have the realisation

$$L_{\alpha+} = -\frac{\partial}{\partial \eta^\alpha}. \quad (6.14)$$

Once again, scaling variables gives us $L_{\alpha+}$ in the momentum representation

$$L_{\alpha+} = -\lambda^{-1} \frac{\partial}{\partial \theta^\alpha}.$$

Lastly, we shall calculate $L_{\mu\alpha}$; let $\rho = e^{\rho^{\mu\alpha} L_{\mu\alpha}}$, and so we have

$$\begin{aligned} \rho^{-1} x &= e^{-\rho^{\mu\alpha} L_{\mu\alpha}} e^{q^\mu J_{\mu+} + \eta^\alpha L_{\alpha+}} e^{\phi \tilde{J}_{+-}} G_0, \\ &= e^{q^\mu J_{\mu+} - [\rho^{\nu\alpha} L_{\nu\alpha}, q^\mu J_{\mu+}]} e^{\eta^\alpha L_{\alpha+} - [\rho^{\nu\alpha} L_{\nu\alpha}, \eta^\beta L_{\beta+}]} e^{\rho^{\mu\alpha} L_{\mu\alpha}} e^{\phi \tilde{J}_{+-}} G_0. \end{aligned}$$

To simplify this expression we use the commutation relations

$$\begin{aligned}
q^\mu \rho^{\nu\alpha} [J_{\mu+}, L_{\nu\alpha}] &= \rho^{\nu\alpha} q_\nu L_{\alpha+}, \\
-\eta^\beta \rho^{\nu\alpha} \{L_{\nu\alpha}, L_{\beta+}\} &= \eta^\beta \rho^{\nu\alpha} \varepsilon_{\alpha\beta} (J_{\nu+} + a \varepsilon_\nu^\mu J_{\mu+}), \\
&= -\rho^{\nu\alpha} \eta_\alpha J_{\nu+} - a \rho^{\mu\alpha} \eta_\alpha \varepsilon_\mu^\nu J_{\mu+}.
\end{aligned}$$

thus we can write

$$\rho f(q, \eta, \phi) = f(q^\mu - \rho^{\mu\alpha} \eta_\alpha - \rho^{\nu\alpha} a \eta_\alpha \varepsilon_\nu^\mu, \eta^\alpha + \rho^{\nu\alpha} q_\nu, \phi),$$

and so

$$\mathcal{L}_{\nu\alpha} = -\eta_\alpha \left(\frac{\partial}{\partial q^\nu} + a \varepsilon_\nu^\mu \frac{\partial}{\partial q^\mu} \right) + q_\nu \frac{\partial}{\partial \eta^\alpha}. \quad (6.15)$$

Once again, changing to momentum representation and re-arranging gives us

$$L_{\mu\alpha} = p_\mu \frac{\partial}{\partial \theta^\alpha} - \theta_\alpha \frac{\partial}{\partial p^\mu} - a \theta_\alpha \varepsilon_\mu^\nu \frac{\partial}{\partial p^\nu}.$$

The full set of generators for the superfield realisation of $D(2, 1; \alpha)$ is therefore

$$\begin{aligned}
J_{\mu+} &= -\lambda^{-1} \frac{\partial}{\partial p^\mu}, \\
L_{\alpha+} &= -\lambda^{-1} \frac{\partial}{\partial \theta^\alpha}, \\
L_{\alpha-} &= \frac{1}{2} \lambda (p^\nu p_\nu + \theta^\beta \theta_\beta) \frac{\partial}{\partial \theta^\alpha} - \theta_\alpha \lambda^2 \frac{\partial}{\partial \lambda} - a \lambda \theta_\alpha p^\mu \varepsilon_\mu^\nu \frac{\partial}{\partial p^\nu}, \\
L_{\mu\alpha} &= p_\mu \frac{\partial}{\partial \theta^\alpha} - \theta_\alpha \frac{\partial}{\partial p^\mu} - a \theta_\alpha \varepsilon_\mu^\nu \frac{\partial}{\partial p^\nu}, \\
K_{\alpha\beta} &= \theta_\alpha \frac{\partial}{\partial \theta^\beta} + \theta_\beta \frac{\partial}{\partial \theta^\alpha}, \\
J &= -p^\mu \varepsilon_\mu^\nu \frac{\partial}{\partial p^\nu} + \frac{a}{1-a^2} \left(\lambda \frac{\partial}{\partial \lambda} - p^\mu \frac{\partial}{\partial p^\mu} - \theta^\alpha \frac{\partial}{\partial \theta^\alpha} \right), \\
J_{+-} &= -\lambda \frac{\partial}{\partial \lambda} - \frac{a^2}{1-a^2} \left(\lambda \frac{\partial}{\partial \lambda} - p^\mu \frac{\partial}{\partial p^\mu} - \theta^\alpha \frac{\partial}{\partial \theta^\alpha} \right), \\
J_{\mu-} &= \frac{1}{2} \lambda \theta^\alpha \theta_\alpha \frac{\partial}{\partial p^\mu} + \frac{1}{2} \lambda a \varepsilon_\mu^\nu \theta^\alpha \theta_\alpha \frac{\partial}{\partial p^\nu} + \frac{1}{2} \lambda \varepsilon_{\mu\nu} \varepsilon_\rho^\sigma p^\nu p^\rho \frac{\partial}{\partial p^\sigma} \\
&\quad - \lambda^2 p_\mu \frac{\partial}{\partial \lambda} + \frac{1}{2} \lambda p_\mu p^\nu \frac{\partial}{\partial p^\nu} \\
&\quad - \frac{\lambda a (a p_\mu + \varepsilon_{\mu\rho} p^\rho)}{1-a^2} \left(\lambda \frac{\partial}{\partial \lambda} - p^\nu \frac{\partial}{\partial p^\nu} - \theta^\alpha \frac{\partial}{\partial \theta^\alpha} \right).
\end{aligned} \quad (6.16)$$

Note that we have included the three previously calculated generators for completeness of the above set.

If we compare this realisation with that obtained for $osp(d, 2/2)$ (see [8] or chapter 4) the similarities are evident (although the realisation of $J_{\mu-}$ requires some attention). In fact, if we allow $\alpha \rightarrow 1$ (and thus $a \rightarrow 0$), which corresponds to $D(2, 1; \alpha) \cong osp(2, 2/2)$, then it can easily be seen that the above relations are in fact identical to those obtained using the standard superfield [8] for the massless case.

Although the commutation relations of the above generators J_{MN} must be equal to those given in (6.8) and (4.17), we shall test this by calculating $\{L_{\mu\alpha}, L_{\nu\beta}\}$ (We could choose any of the relations).

Equation (6.8) tells us that $\{L_{\mu\alpha}, L_{\nu\beta}\} = \varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}\tilde{J} - \eta_{\mu\nu}K_{\alpha\beta}$. Using the definition of $L_{\mu\alpha}$ given in (6.15) gives

$$\begin{aligned} & \{-\eta_\alpha\partial_\mu - a\eta_\alpha\varepsilon_\mu^\nu\partial_\sigma + q_\mu\partial_\alpha, -\eta_\beta\partial_\nu - a\eta_\beta\varepsilon_\nu^\rho\partial_\rho + q_\nu\partial_\beta\} \\ &= -\{\eta_\alpha\partial_\mu, q_\nu\partial_\beta\} - a\{\eta_\alpha\varepsilon_\mu^\sigma\partial_\sigma, q_\nu\partial_\beta\} - \{q_\mu\partial_\alpha, \eta_\beta\partial_\nu\} - a\{q_\mu\partial_\alpha, \eta_\beta\varepsilon_\nu^\rho\partial_\rho\}. \end{aligned}$$

Note that the first and third terms are identical, except for the indices, as are the second and fourth terms. Using the identity

$$\{AB, CD\} = \frac{1}{2}\{A, C\}\{B, D\} + \frac{1}{2}[A, C][B, D],$$

where $[A, B] = [C, D] = 0$, we have that the first term is

$$-\frac{1}{2}\eta_{\mu\nu}(2\eta_\alpha\partial_\beta - \varepsilon_{\alpha\beta}) - \frac{1}{2}\varepsilon_{\alpha\beta}(2q_\nu\partial_\mu + \eta_{\mu\nu}).$$

And so terms one and three sum to

$$\varepsilon_{\alpha\beta}(q_\mu\partial_\nu - q_\nu\partial_\mu) - \eta_{\mu\nu}(\eta_\alpha\partial_\beta + \eta_\beta\partial_\alpha).$$

In a similar fashion, we get that the second and fourth terms sum to

$$-a\varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}\eta^\gamma\partial_\gamma - a\varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}q^\rho\partial_\rho.$$

Combining these two expressions together we get

$$\begin{aligned} \{L_{\mu\alpha}, L_{\nu\beta}\} &= \varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}(-q^\rho\varepsilon_\rho^\sigma\partial_\sigma - aq^\rho\partial_\rho - a\eta^\gamma\partial_\gamma) - \eta_{\mu\nu}(\eta_\alpha\partial_\beta + \eta_\beta\partial_\alpha), \\ &= \varepsilon_{\mu\nu}\varepsilon_{\alpha\beta}\tilde{J} - \eta_{\mu\nu}K_{\alpha\beta}, \end{aligned} \tag{6.17}$$

as claimed.

6.2.1 Physical States

The BRST operator for the $D(2, 1; \alpha)$ model can be implemented in a similar way to that seen in chapters 4.4 and 5.4, *i.e.* by considering two linearly independent spinors η^α and η'^α which obey the condition $\eta^\alpha \eta'_\alpha = 1$. An example of two such spinors is, of course,

$$\eta^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \eta'^\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We define the ghost number operator in the usual way,

$$N_{\text{gh}} = \eta^\alpha \eta'^\beta K_{\alpha\beta} = (\eta \cdot \theta)(\eta' \cdot \partial) + (\eta' \cdot \partial)(\eta \cdot \partial), \quad (6.18)$$

where $\partial_\alpha = \partial/\partial\theta^\alpha$. Similarly we can define the BRST operator as

$$\begin{aligned} \Omega &= \eta^\alpha L_{\alpha-}, \\ &= \eta^\alpha \left(\frac{1}{2} \lambda (p^\nu p_\nu + \theta^\beta \theta_\beta) \frac{\partial}{\partial\theta^\alpha} - \theta_\alpha \lambda^2 \frac{\partial}{\partial\lambda} - a \lambda \theta_\alpha p^\mu \varepsilon_\mu{}^\nu \frac{\partial}{\partial p^\nu} \right). \end{aligned} \quad (6.19)$$

The physical states can be calculated in the standard way, *i.e.* considering the effect of Ω on a superfield

$$\psi = A + \theta^\alpha \chi_\alpha + \frac{1}{2} \theta^\alpha \theta_\alpha B \quad (6.20)$$

(for more detail see section 4.4). Explicitly we can write

$$\begin{aligned} \Omega\psi &= \frac{1}{2p_+} p \cdot p (\eta \cdot \chi) + \eta^\alpha \theta_\alpha \left(\frac{1}{2p_+} B + \left[\frac{\partial}{\partial p_+} - \frac{a}{p_+} (p^\mu \varepsilon_{\mu\nu} \partial_{p_\mu}) \right] A \right) \\ &\quad + \frac{1}{2} \theta^\alpha \theta_\alpha \left(-\frac{\partial}{\partial p_+} + \frac{1}{p_+} (1 + a p^\mu \varepsilon_{\mu\nu} \partial_{p_\nu}) \right) (\eta \cdot \chi), \end{aligned} \quad (6.21)$$

and imposing the conditions

$$\Omega\psi = 0, \quad \psi \neq \Omega\psi' \text{ and } N_{\text{gh}}\psi = \ell\psi,$$

for the maximal eigenvalue $\ell = 1$ (in accordance with section 4.4).

Comparing equations (6.20) and (6.21) we see that the components A, χ_α and B of ψ are defined up to addition of functions corresponding to coefficients in (6.21). Imposing the first condition above, we see that the $\eta^\alpha \theta_\alpha$ coefficient

determines simply the p_+ -dependence of A in terms of some unknown B (which is itself determined up to a p_+ -derivative of some function).

Looking at $\eta^\alpha \chi_\alpha$, we find, respectively, from $\mathcal{O}(\theta^0)$ and $\mathcal{O}(\theta^2)$

$$\frac{1}{2p_+}(p \cdot p)(\eta^\alpha \chi_\alpha) = 0, \quad (6.22)$$

$$\left(\frac{\partial}{\partial p_+} - \frac{1 + ap^\mu \varepsilon_{\mu\nu} \partial^\nu}{p_+} \right) (\eta^\alpha \chi_\alpha) = 0. \quad (6.23)$$

We can write p^μ in component form as

$$\begin{aligned} p_R^\mu &= \frac{1}{\sqrt{2}}(p^\mu + \varepsilon^\mu{}_\nu p^\nu), \\ p_L^\mu &= \frac{1}{\sqrt{2}}(p^\mu - \varepsilon^\mu{}_\nu p^\nu). \end{aligned}$$

Assuming that $\eta^\alpha \chi_\alpha = \phi(p)$, where $\phi(p)$ is a function of the form

$$\phi(p) = p_+ \varphi((1 + a \ln p_+)p_R, (1 - a \ln p_+)p_L),$$

we have

$$\frac{\partial}{\partial p_+} \phi(p) = \frac{1}{p_+} \left(1 + a(p^\mu \varepsilon_\mu{}^\nu \frac{\partial}{\partial p^\nu}) \right) \phi,$$

as in (6.23). Hence the given form of $\phi(p)$ solves equation (6.23).

Enforcing (6.22) gives that $\phi(p)$ satisfies $p \cdot p = 0$, or

$$\frac{1}{2p_+} p_R \cdot p_L p_+ \varphi(\zeta^+ p_R, \zeta^- p_L) = 0,$$

where $\zeta^\pm = 1 \pm a \ln p_+$. Equivalently, in Fourier space the constraints are solved by the physical states

$$\varphi(x_R, x_L) \equiv \int (\chi \cdot \psi) e^{-ip \cdot x} dx,$$

that satisfy

$$\frac{\partial}{\partial x_R} \frac{\partial}{\partial x_L} \varphi \left(\frac{x_R}{\zeta^+}, \frac{x_L}{\zeta^-} \right) = 0. \quad (6.24)$$

Moreover, if we assume $H\psi = 0$ (H the Hamiltonian) on the physical states and we assume the Schrödinger equation

$$H = -i \frac{d}{d\tau},$$

then these functions are independent of τ .

Finally, by once again employing the triviality of $Im\Omega$ for the maximal ghost number we can show that the physical state is unique, as there is no function ψ' such that $\psi = \Omega\psi'$ lies in the same cohomology as ψ .

The system we have constructed can be interpreted as the ‘quantisation’ of a classical ‘ $D(2, 1; \alpha)$ ’ particle. The p_+ -dependence on physical wavefunctions provides indirect evidence that the model involves a more subtle implementation of diffeomorphisms than usual. Note that the two-dimensional case has the unique property that Lorentz invariance is not broken. The metric $x^\mu x^\nu \eta_{\mu\nu} = x_R x_L$ is still a world-line scalar if x_R, x_L transform as densities:

$$x'_{R,L}(\tau') d\tau'^{\pm a} = x_{R,L}(\tau) d\tau^{\pm a}.$$

Corresponding covariant actions may be responsible (after gauge fixing) for the p_+ -scaling behaviour*.

6.3 Classical Hamiltonian and action

In the previous section we did not explicitly calculate the corresponding Hamiltonian $H = -\{\mathcal{F}, \Omega\}$, nor did we specify a gauge fixing function \mathcal{F} . The reason behind this is simple, as we have no classical model with which to compare our quantised particle we do not have any guide as to what our quantised Hamiltonian should look like, and thus no guide as to which gauge fixing function \mathcal{F} we should choose. In this section we postulate an \mathcal{F} which leads to an acceptable looking Hamiltonian, and from there derive a classical action S . This is the action which defines the classical system which corresponds to the quantum system derived from the algebraic structure in section 6.2.

By definition the gauge fixing function \mathcal{F} is Grassmann odd and has ghost number -1 , thus it obeys the equation

$$[[N_{\text{gh}}, \mathcal{F}]] = -\mathcal{F}. \quad (6.25)$$

*The gauge equivalence class of λ , or e , namely $\int_{\tau_i}^{\tau_f} e(\tau) d\tau$, is proportional to λ in the present case $\dot{\lambda} = 0$

As well as these constraints on \mathcal{F} , in the $D(2, 1; \alpha)$ system we must make sure that Hamiltonian generated is general enough to encompass the extended behaviour of the system (as compared with the corresponding $osp(2, 2/2)$ system) and that in the limit $\alpha \rightarrow 1$ it reduces to Hamiltonian for the $osp(2, 2/2)$ system of chapter 4.

Firstly we express the ghost number operator (see (6.18)) as

$$N_{\text{gh}} = \frac{1}{2}(K_{22} - K_{11}) = \theta_2 \partial_2 - \theta_1 \partial_1.$$

As a first guess at the gauge fixing function we choose $\mathcal{F} = \eta'^\alpha \theta_\alpha$. This function has ghost number -1 and is Grassmann odd, checking that it satisfies (6.25) we get

$$[[N_{\text{gh}}, \mathcal{F}]] = \frac{1}{\sqrt{2}}\{\theta_2 \partial_2 - \theta_1 \partial_1, \theta_2 - \theta_1\} = -\frac{1}{\sqrt{2}}(\theta_2 - \theta_1) = -\mathcal{F},$$

as we desire. However this \mathcal{F} falls down when we generate the corresponding Hamiltonian, which is found to be independent of α . Thus the Hamiltonian generated by this gauge fixing function cannot reproduce the α dependent quantised system of section 6.2.

Our second choice for the gauge fixing function is $\mathcal{F} = \eta'^\alpha \partial_\alpha$. This function also satisfies the necessary conditions however it once again falls down when we generate the Hamiltonian, as this H does not revert to the $osp(2, 2/2)$ Hamiltonian as $\alpha \rightarrow 1$. Thus we are led to choosing our gauge fixing function as a scalar combination of the two given above, *i.e.*

$$\mathcal{F} = \frac{1}{\sqrt{2}}((\theta_2 - \theta_1) + b(\partial_1 - \partial_2)), \quad (6.26)$$

where b is an arbitrary scalar, and we have changed the overall sign of the second term. This \mathcal{F} is Grassmann odd, has ghost number -1 and obeys equation (6.25). The corresponding Hamiltonian can now be calculated

$$\begin{aligned} H &= [[\Omega, \mathcal{F}]] = [[\Omega, \mathcal{F}_\theta - b\mathcal{F}_\partial]] \\ [[\Omega, \mathcal{F}_\theta]] &= \left[\left[\frac{1}{2}\lambda p^2(\partial_1 + \partial_2) + \frac{1}{2}\lambda\theta^\beta\theta_\beta(\partial_1 + \partial_2), \theta_2 - \theta_1 \right] \right], \\ &= -\frac{1}{2}\lambda(p^2 + \theta^\beta\theta_\beta), \end{aligned}$$

and

$$\begin{aligned} \llbracket \Omega, -\mathcal{F}_\partial \rrbracket &= \frac{1}{2} \llbracket -(\theta_1 + \theta_2)\lambda^2\partial_\lambda - a\lambda(\theta_1 + \theta_2)p^\mu\varepsilon_\mu^\nu\partial_\nu + \lambda\theta^\beta\theta_\beta(\partial_1 + \partial_2), \partial_a - \partial_2 \rrbracket, \\ &= \lambda^2\partial_\lambda + a\lambda p^\mu\varepsilon_\mu^\nu\partial_\nu + \frac{1}{2}\lambda(\theta_1 - \theta_2)(\partial_1 + \partial_2). \end{aligned}$$

Thus the Hamiltonian for the $D(2, 1; \alpha)$ system is

$$H = -\frac{1}{2}\lambda(p^2 + \theta^\beta\theta_\beta) + b\lambda^2\partial_\lambda + ab\lambda p^\mu\varepsilon_\mu^\nu\partial_\nu + \frac{1}{2}b\lambda(\theta_1 - \theta_2)(\partial_1 + \partial_2), \quad (6.27)$$

where $\lambda = 1/p_+$. Notably this action is general enough so as to encompass the special behaviour of the $D(2, 1; \alpha)$ system (as $a = a(\alpha)$) and can reduce to the Hamiltonian for the massless scalar particle in the $osp(2, 2/2)$ case of chapter 4.

Having now derived the Hamiltonian of the $D(2, 1; \alpha)$ system we now seek to calculate the corresponding classical action and Lagrangian \mathcal{L} . We do this by means of a Legendre transformation and the Hamiltonian equations of motion. This process is the reverse of that given by equations (2.5) to (2.6). Given the Hamiltonian, we can write the Lagrangian as

$$\mathcal{L} = \sum \dot{q}p - H(q, p), \quad (6.28)$$

where q, p are the generalised co-ordinates of H . \dot{q} is calculated by means of the Hamiltonian equations of motion, seen in equation (2.7);

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial p_\mu} = \lambda(-p^\mu + ab\varepsilon_\nu^\mu x^\nu), \\ \dot{\lambda} &= \frac{\partial H}{\partial \partial_\lambda} = b\lambda^2, \\ \dot{\theta}^\alpha &= \frac{\partial H}{\partial \partial_\alpha} = \frac{1}{\sqrt{2}}\lambda b(\theta_1 - \theta_2)\eta^\alpha\partial_\alpha, \\ \therefore \dot{\theta}^\alpha\partial_\alpha &= \frac{1}{2}\lambda b(\theta_1 - \theta_2)(\partial_1 + \partial_2). \end{aligned} \quad (6.29)$$

Note that here we have used

$$\eta^\alpha\partial_\alpha = \frac{1}{\sqrt{2}}(\partial_1 + \partial_2).$$

From the first of these equations we can write

$$p_\mu = -\frac{\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu,$$

and so, by substituting the above expressions into (6.28), we get

$$\begin{aligned}\mathcal{L} = & \dot{x}^\mu \left(-\frac{\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu \right) + b\lambda^2\partial_\lambda + \frac{1}{2}\lambda b(\theta_1 - \theta_2)(\partial_1 + \partial_2) \\ & + \frac{1}{2}\lambda \left(\left(\frac{-\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu \right) \left(\frac{-\dot{x}^\mu}{\lambda} - ab\varepsilon^{\mu\nu}x_\nu \right) + \theta^\beta\theta_\beta \right) - b\lambda^2\partial_\lambda \quad (6.30) \\ & - ab\lambda \left(\frac{\dot{x}^\mu}{\lambda} - ab\varepsilon_\nu^\mu x^\nu \right) \varepsilon_\mu^\nu x_\nu - \frac{1}{2}b\lambda(\theta_1 - \theta_2)(\partial_1 + \partial_2).\end{aligned}$$

Thus the Lagrangian can be written

$$\mathcal{L} = -\frac{1}{2}\frac{\dot{x}^2}{\lambda} - \lambda\frac{1}{2}(ab)^2x^2 - ab\varepsilon_{\mu\nu}\dot{x}^\mu x^\nu + \frac{1}{2}\lambda\theta^\beta\theta_\beta. \quad (6.31)$$

The classical Lagrangian (and action) corresponding to the quantum Hamiltonian (6.27) should be free of ghosts (as they only arise in the extended phase space of the BFV-BRST construction). Likewise the canonical momentum conjugate to the Lagrange multiplier λ should not be present. Thus we arrive at the classical action of the $D(2, 1; \alpha)$ system by decoupling the ghost sector from the action above, *i.e.* only considering the bosonic part

$$S = \int_{\tau_i}^{\tau_f} d\tau \left[-\frac{1}{2}\frac{\dot{x}^2}{\lambda} - \lambda\frac{1}{2}(ab)^2x^2 - ab\varepsilon_{\mu\nu}\dot{x}^\mu x^\nu \right]. \quad (6.32)$$

By comparing this with the action given in equation (4.5) we can see that (6.32) corresponds to a massless scalar particle in a potential well. In fact if we ignore the last term in (6.32) then we have arrived at the classical action of an oscillating massless particle (*i.e.* where the potential is proportional to x^2). For further details of the oscillator in the classical or quantum case see [12, 69]. The final term of (6.32) introduces a cross term between velocity and position. Comparing this term with the potential term in equation (4.16), we see that the cross term is similar to that produced by a homogeneous electromagnetic field $F_{\mu\nu} = ab\varepsilon_{\mu\nu}$.

The action (6.32) also satisfies the condition that as $a \rightarrow 0$ (which is equivalent to $\alpha \rightarrow 1$), \mathcal{L} becomes the Lagrangian of the massless scalar particle. The constant b is also important as it distinguishes between the parts of the action that arise from each of the two gauge fixing functions we tried earlier; \mathcal{F}_θ and \mathcal{F}_∂ . We can now set $b = 1$ without affecting the behaviour of the particle.

For the sake of completeness, we shall identify the total covariant energy and angular momentum of the classical $D(2, 1; \alpha)$ particle, as well as calculating the Euler-Lagrange equations of motion. The total covariant energy is given by

$$\begin{aligned} P_\mu &= \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \\ &= -\frac{\dot{x}_\mu}{\lambda} - ab\varepsilon_{\mu\nu}x^\nu. \end{aligned}$$

The total angular momentum is

$$\begin{aligned} M_{\mu\nu} &= P_\mu x_\nu - P_\nu x_\mu, \\ &= \frac{1}{\lambda}(\dot{x}_\nu x_\mu - \dot{x}_\mu x_\nu) + ab(\varepsilon_{\nu\rho}x_\mu - \varepsilon_{\mu\rho}x_\nu)x^\rho. \end{aligned} \quad (6.33)$$

The Euler Lagrange equations of motion are (see (2.2))

$$-\frac{\ddot{x}_\mu}{\lambda} + (ab)^2\lambda x_\mu - 2ab\varepsilon_{\mu\nu}\dot{x}^\nu = 0, \quad (6.34)$$

$$\frac{\dot{x}^2}{2\lambda^2} - (ab)^2x^2 = 0. \quad (6.35)$$

By virtue of the fact that the Lagrangian is independent of $\dot{\lambda}$, the second of these two equations is actually a constraint on the system. Hence, the Hessian of the Lagrange function vanishes identically, except for its components $\partial^2 \mathcal{L}/(\partial \dot{x}^\mu \partial \dot{x}^\nu)$ [14], thereby illustrating the singular nature of the action (6.32). Thus we can identify (6.35) as a first class constraint of the $D(2, 1; \alpha)$ system.

Now that we have determined the classical action corresponding to the $D(2, 1; \alpha)$ particle it is possible to start the loop again, so to speak. That is, identify the first class constraints, extend the phase space and follow the BFV-BRST quantisation procedure (as was done for the scalar particle in chapter 4 and the spinning particle in chapter 5) in order to arrive at the quantised $D(2, 1; \alpha)$ particle. However we shall not do this; given that we correctly choose the gauge fixing condition (see (4.7) for the scalar particle condition) we would end up with exactly the same quantised system as system as that obtained in section 6.2. Secondly, the aim in this chapter (and throughout this thesis) is to study the algebras of quantisation, which we have already done for the $D(2, 1; \alpha)$ particle in the previous section.

6.4 Conclusion

In this chapter we have shown that it is possible to extend the BFV-BRST quantisation algebra $iosp(d, 2/2)$ in two dimensions into the more general classical simple Lie superalgebra $D(2, 1; \alpha)$. To do this we started without a classical physical model of a particle, and so relied entirely on the algebraic structure as our guide. In section 6.2 we showed that the algebraic model that we had constructed was an admissible quantisation superalgebra, and so provided a quantisation of the corresponding classical system. In section 6.3 we then calculated the classical action corresponding to the quantum system. If this action was used as a starting point, then the BFV-BRST process, as outlined in chapter 3, could be followed and the quantum system of 6.2 would be derived.

An alternative (and equally valid) method of presenting this chapter would have been to start with the classical action (6.32) and from there proceed to quantise the system and demonstrate that it obeys a $D(2, 1; \alpha)$ quantisation superalgebra.

Chapter 7

Two-time physics

This chapter contains some unfinished work and suggestions of future directions in a potentially exciting new area of mathematical physics. This area is two-time physics, as proposed by Bars and collaborators [70, 71, 72, 73, 74, 75, 76, 77, 4, 78, 79]. Two-time physics, as the name implies, is a theory based upon the assumption that ordinary physical theories can be formulated with two independent times without the need for the introduction of ghosts. By making various gauge choices in the two-time formulation one can produce an infinite number of sectors of one-time physics [74, 75, 76], for example the free relativistic particle, with or without mass, the hydrogen atom, the harmonic oscillator, particles in various curved space-times and particles in general potentials $V(r)$. The theory can also be extended to M theory [78] and strings [72, 79]. This broad sweep is possible due to the infinite number of ways of making gauge choices in two-time theory that lead to a definition of physical time in one-time. Each of these choices is connected through an $Sp(2, \mathbb{R})$ duality.

In section 7.1 we shall review the theory of two-time physics, as well as detailing the construction necessary to reveal its presence. In the next section we follow through the BRST formalism of chapter 3 in order to produce a BRST equivalent form of a two-time particle. This section contains incomplete work, but should prove a useful starting point and guide for anyone wishing to take up this challenge. In the final section we shall discuss possible future directions and

tests for two-time physics, from a BFV-BRST perspective.

7.1 Construction of two-times

The formalism for two-time physics for a particle $X^m(\tau)$ is a simple $Sp(2, \mathbb{R})$ gauge theory; the commutation relation $[x, p] = i$ admits an automorphism symmetry of $Sp(2, \mathbb{R})$, and treats x, p as a doublet. The basic idea is to turn this global symmetry of quantum mechanics into a local symmetry of a theory. This is the opposite of BRST theory, where the local gauge invariance of a constrained Hamiltonian system associated with the first class constraints is replaced by a global symmetry generated by the BRST charge (see section 3.1.2). The three-parameter local symmetry $Sp(2, \mathbb{R})$ includes a τ -reparametrisation as one of the local transformations, and therefore can be regarded as a generalisation of gravity on the world-line. Crucially, the $Sp(2, \mathbb{R})$ gauge theory is non-trivial and physically consistent only if the particle has two time-like coordinates $X^0(\tau), X^{0'}(\tau)$, and has a global symmetry $SO(d, 2)$, which is interpreted as a Lorentz group with two times.

The $SO(d, 2)$ group is realised in the same unique representation for diverse one-time physical systems, with the same eigenvalues of the Casimir operators. Each one-time system provides a different basis with the same representation of $SO(d, 2)$.

The group $Sp(2, \mathbb{R})$ treats the canonically conjugate coordinates (x, p) as a doublet in phase space (see section 3.3). A consequence of this $Sp(2, \mathbb{R})$ gauge theory is that duality and two-times are inextricably connected. In fact, the local $Sp(2, \mathbb{R})$ symmetry requires one extra time-like coordinate, plus one extra spacelike coordinate to lift a system from one-time physics to its most symmetric $SO(d, 2)$ covariant form in two-times. This requirement of extra dimensions to exhibit a higher symmetry can be seen in other observations involving duality, for example in M-theory [80, 4, 79].

We shall now review the construction of two-time physics for a relativistic

particle, given by Bars *et. al.* in [74]. In the next section we shall slightly modify this construction so as to suit our purposes, but for now we shall present it in its original form. Consider $X^M(\tau)$ and $P^M(\tau)$ as a canonically conjugate pair of operators. We relabel these coordinates as

$$X_i^M(\tau) = (X_1^M(\tau), X_2^M(\tau)) = (X^M(\tau), P^M(\tau)).$$

The $Sp(2, \mathbb{R})$ symmetry acts as

$$\delta_w X_i^M(\tau) = \varepsilon_{ik} w^{kl}(\tau) X_l^M(\tau),$$

where w^{ij} is symmetric and contains the three local parameters of $Sp(2, \mathbb{R})$. The $Sp(2, \mathbb{R})$ gauge field $A^{ij}(\tau)$ is symmetric and transforms in the standard way:

$$\delta_w A^{ij} = \partial_\tau w^{ij} + w^{ik} \varepsilon_{kl} A^{lj} + w^{jk} \varepsilon_{kl} A^{il}.$$

The covariant derivative can be written

$$D_\tau X_i^M(\tau) = \partial_\tau X_i^M + \varepsilon_{ik} A^{kl} X_l^M. \quad (7.1)$$

Note that here we have changed the sign of the gauge field term from that which Bars used, to coincide with the more standard definition of ε^{ij} , *i.e.* $\varepsilon_{ij} \varepsilon^{jk} = -\delta_i^k$.

An action that is invariant under $Sp(2, \mathbb{R})$ gauge symmetry [74] is

$$S = \int_0^\tau d\tau \mathcal{L} = \frac{1}{2} \int_0^\tau d\tau (D_\tau X_i^M) \varepsilon^{ij} X_j^N \eta_{MN}, \quad (7.2)$$

$$= \int_0^\tau d\tau \left(\partial_\tau X_1^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN}, \quad (7.3)$$

where we have dropped a total derivative. In the second of these two expressions, (7.3), X_1^M and X_2^N are canonical conjugates, *i.e.*

$$X_i^M = \frac{\partial \mathcal{L}}{\partial X_M^j}, \quad i \neq j. \quad (7.4)$$

This is consistent with the assignment of X_1^M, X_2^N with X^M, P^N , however, as shall be revealed in the next section, the situation is not quite so straightforward.

The global symmetry of the action leaves the metric invariant under global Lorentz transformations $SO(d, 2)$. The generators of this symmetry are

$$L^{MN} = \varepsilon^{ij} X_i^M X_j^N = X^M P^N - X^N P^M. \quad (7.5)$$

The L^{MN} are gauge invariant under $Sp(2, \mathbb{R})$. The full physical information of the theory is contained in the gauge invariant L^{MN} .

The equations of motion which follow from the Lagrangian \mathcal{L} yield the first class constraints

$$X^M X^N \eta_{MN} = X^M P^N \eta_{MN} = P^M P^N \eta_{MN} = 0. \quad (7.6)$$

If the signature of the metric η_{MN} has only one time-like coordinate, then the only classical solution to these constraints is that X^M, P^N are parallel and light-like. This is trivial in that the angular momentum is zero. Allowing the metric to have more than two time-like coordinates leads to the introduction of ghosts and so are ruled out by the premise that the construction should be free of ghost particles. Thus the only possible non-trivial solutions are provided when η_{MN} has two time-like coordinates, *i.e.* two-time physics.

Using the constraints (7.6), it is straightforward to show that all Casimir operators of $SO(d, 2)$ vanish at the classical level. In contrast the quantised Casimirs of $SO(d, 2)$, after having taken quantum ordering into account, are not zero. The quadratic Casimir of $Sp(2, \mathbb{R})$ vanishes, and is related to $C_2(SO(d, 2))$. This fact can be used to fix the quantised Casimirs as [74, 4]

$$C_2(Sp(2)) = 0 \quad \left\{ \begin{array}{l} C_2(SO(d, 2)) = 1 - \frac{d^2}{4}, \\ C_3(SO(d, 2)) = \frac{d}{3!} \left(1 - \frac{d^2}{4} \right) \end{array} \right. \quad (7.7)$$

This result implies that the diverse one-time physical models that emerge by gauge fixing must have precisely zero $SO(d, 2)$ Casimir eigenvalues at a classical level, and some non-trivial Casimirs that label physical Hilbert space in the first quantised version. This in turn suggests that the free relativistic massless particle in $(d - 1)$ space dimensions should have a Hilbert space dual to that of a particle in the $1/r$ potential in the same dimensions. Apart from a choice of basis, they should also share the same unique $SO(d, 2)$ representation.

The system, as described above, is the basis for a two-time particle. We now describe how diverse one-time physical systems emerge from the same two-time theory, by taking various appropriate gauge choices that embed physical time in

different ways in the extra dimensions [74, 75, 78, 79]. $Sp(2, \mathbb{R})$ has three gauge parameters, and so we have the freedom to fix three functions, this is done as follows.

- Make n gauge choices ($n = 2, 3$) for the $2d + 4$ functions X^M, P^M .
- Solve n constraints, which determines n additional functions, thus obtaining a gauge fixed configuration of X_0^M, P_0^M parameters in terms of $2(d + 2 - n)$ independent functions $x(\tau), p(\tau)$.
- The dynamics for the remaining degrees of freedom x, p are determined by inserting $X_0^M(\tau), P_0^M(\tau)$ into the original action (7.3), thus constructing a new one-time physics action

$$S(x, p) = S_0(X_0^M, P_0^M) = \int_0^\tau d\tau \mathcal{L}(x(\tau), p(\tau), A(\tau)).$$

In the above equation $A(\tau)$ is the remaining gauge potential if $n = 2$, but is absent if $n = 3$.

The one-time physical system that emerges from this process is recognised by studying the form of the Lagrangian \mathcal{L} .

The action $S(x, p)$ inherits the $SO(d, 2)$ symmetry, which however is now realised non-linearly [74]. The presence of this hidden symmetry was not suspected for most of the diverse physical systems constructed by Bars *et. al.*. The original generators of the symmetry L^{MN} in (7.5) are gauge invariant and therefore must be generators of the symmetry of the new action. The symmetry generators for the new action can be constructed explicitly at any τ by inserting $X_0^M(\tau), P_0^M(\tau)$ into (7.5). These L^{MN} do indeed [74] form the algebra $SO(d, 2)$ under Poisson brackets

$$\{L^{MN}, L^{RS}\} = \eta^{MR}L^{NS} + \eta^{NS}L^{MR} - \eta^{NR}L^{MS} - \eta^{MS}L^{NR}. \quad (7.8)$$

If τ appears explicitly (as is the case for $n = 3$) it is treated as a parameter in the Poisson brackets. The symmetry transformations of the canonical coordinates

$\delta x(\tau), \delta p(\tau)$ can be obtained through

$$\delta x(\tau) = \frac{1}{2}\varepsilon_{MN}\{L^{MN}(\tau), x(\tau)\}, \quad \delta p(\tau) = \frac{1}{2}\varepsilon_{MN}\{L^{MN}(\tau), p(\tau)\},$$

where once again we are treating τ as a parameter.

The one-time physical system described by $\mathcal{L}(x, p)$ can be first quantised in the usual way (Dirac quantisation). Once this is done we may then construct the quantum generators of $SO(d, 2)$ from the classical ones (7.5). In the Hamiltonian formalism, we take $\tau = 0$ and order operators to insure $L^{MN}(x, p)$ is Hermitian. Bars found he needed corrections [74] to some of the L^{MN} , *i.e.* he included anomaly terms to ensure closure of $SO(d, 2)$. Once this is achieved in some fixed gauge we have the physical space for a corresponding physical system described by a representation of the $SO(d, 2)$ algebra. This has been found to agree with the representation obtained using covariant quantisation in every case studied by Bars [4, 78, 79].

Examples of the application of this method to free particles, the Hydrogen atom, harmonic oscillator, Anti-de Sitter symmetry, M branes and String theory can be found in [75, 76, 78, 79].

7.2 BRST construction in two-time physics

Two-time physics, as developed by Bars, deals with the formulation of physical theories in quantum mechanical form. He has not, however, studied in depth the question of quantisation of a two-time system. In this section we seek to quantise the general two-time system by following the BRST construction given in section 3.1. This involves the introduction of ghosts associated with each first class constraint, construction of the BRST operator and finally identification of an appropriate Hamiltonian. We then quantise the system and perform a β -limiting process in an attempt to arrive at an identical solution to that given by Bars (see [4] for a review of much of Bars' work). The success of this process would provide further affirmation of the validity of two-time physics.

Our starting point is the action given in equation (7.2)

$$S = \frac{1}{2} \int_0^\tau d\tau (D_\tau X_i^M(\tau)) \varepsilon^{ij} X_j^N(\tau) \eta_{MN}.$$

In order to be as general as possible, we do not define the range of M, N , except to say that the constant metric η_{MN} is symmetric. The covariant derivative can be rewritten using (7.1). Using this, and dropping the total derivative, Bars arrives at the equation for the action given in (7.3). This then leads to $X_1(\tau)$ and $X_2(\tau)$ being identified as canonical conjugates. In our treatment we similarly drop the total derivative, but write the action in a more general form as

$$S = \frac{1}{2} \int_0^\tau d\tau ((\partial_\tau X_i^M) X_j^N \varepsilon^{ij} \eta_{MN} + \varepsilon_{ik} \varepsilon^{ij} A^{kl} X_l^M X_j^N \eta_{MN}).$$

Using Bars' definition of $\varepsilon^{12} = 1$, we can write the Lagrangian as

$$\mathcal{L} = \frac{1}{2} \partial X_i^M \varepsilon^{ij} X_j^N \eta_{MN} - \frac{1}{2} A^{ij} X_i^M X_j^N \eta_{MN}. \quad (7.9)$$

The canonically conjugate momenta $P_M^i(\tau)$ of the coordinates $X_i^M(\tau)$ can be calculated and are

$$P_M^i = \frac{\partial \mathcal{L}}{\partial \dot{X}_i^M} = \frac{1}{2} \varepsilon^{ij} X_j^N \eta_{MN}. \quad (7.10)$$

This is notably different from equation (7.4), which is the corresponding expression derived by Bars *et. al.* [70, 74, 4]. In his work X_1 and X_2 were canonically conjugate, but here we can see that $P^1 = -1/2 X^2$, $P^2 = 1/2 X^1$.

The equations of motion for the system can be easily determined using the Euler-Lagrange equation. Corresponding to the X_i^M are the equations

$$\begin{aligned} (\dot{X}_2^N + A^{11} X_1^N + A^{12} X_2^N) \eta_{MN} &= 0, \\ (-\dot{X}_1^N + A^{22} X_2^N + A^{12} X_1^N) \eta_{MN} &= 0. \end{aligned}$$

Whilst corresponding to the A^{ij} are the equations

$$\begin{aligned} A^{11} &: X_1^M X_1^N \eta_{MN} = 0, \\ A^{12} &: X_1^M X_2^N \eta_{MN} = 0, \\ A^{22} &: X_2^M X_2^N \eta_{MN} = 0, \end{aligned} \quad (7.11)$$

These last three equations form the first class constraints of the system.

The BRST-extended phase space necessary for BFV quantisation of the system consists of the coordinates X_1^M, X_2^M , along with their canonically conjugate momenta P_M^1, P_M^2 respectively. As well, we have the ‘gauge field’ variable A^{ij} and its associated momentum ${}_A\pi_{ij}$, where ${}_A\pi_{ij} = \partial\mathcal{L}/\partial\dot{A}^{ij} = 0$. The first class constraints for the system can be written

$$\phi_{ij} = X_i^M X_j^N \eta_{MN} = 0 \text{ and } {}_A\phi_{ij} = {}_A\pi_{ij} = 0.$$

The definition of the momentum P_M^i given in (7.10) is problematic. The constraint that it places on P^i and X_i leads to non-weakly-vanishing Poisson brackets with ϕ_{ij} and so is a second class constraint. The BRST construction can only be formulated for first class constraints, and so the presence of these second class constraints is a significant complication; the system must be reformulated in such a manner so as to contain only first class constraints. The first attempt at removing second class constraints was carried out by Dirac [1] and was discussed in section 2.1.1. We shall follow this procedure here, and in section 7.3 we shall briefly discuss alternative methods.

Following the procedure developed in section 2.1.1 we label the second class constraints by

$$\chi_{kM} = P_{kM} + \frac{1}{2}X_{kM} = 0, \quad (7.12)$$

where the lower case indices i, j, k, l run over the range $(1, 2)$, whilst the upper case M, N run over the unspecified range with metric η_{MN} . We now form the matrix of Poisson brackets of second class constraints

$$\begin{aligned} \Delta_{kM, lN} &= \{\chi_{kM}, \chi_{lN}\}, \\ &= \frac{1}{2}(\{P_{kM}, \chi_{lN}\} + \{X_{kM}, P_{lN}\}), \end{aligned} \quad (7.13)$$

$$= \frac{1}{2}(\varepsilon_{kl}\eta_{MN} - \varepsilon_{lk}\eta_{NM}) = \varepsilon_{kl}\eta_{MN}. \quad (7.14)$$

The inverse $C = \Delta^{-1}$ of this matrix is easily calculated

$$C^{kM, lN} = \varepsilon^{kl}\eta^{MN}.$$

We can now calculate the Dirac brackets explicitly

$$\begin{aligned}
\{X_{kM}, X_{lN}\}_D &= \{X_{kM}, X_{lN}\} - \sum \{X_{kM}, \chi_{k'M'}\} C^{k'M'l'N'} \{\chi_{l'N'}, X_{lN}\}, \\
&= 0 - \sum \{x_{kM}, P_{k'M'}\} C^{k'M'l'N'} \{P_{l'N'}, X_{lN}\}, \\
&= +\varepsilon_{kk'} \eta_{MM'} \varepsilon^{k'l'} \eta^{M'N'} \varepsilon_{l'l'} \eta_{NN'} = \varepsilon_{kl} \eta_{MN}.
\end{aligned} \tag{7.15}$$

$$\begin{aligned}
\{X_{kM}, P_{lN}\}_D &= \varepsilon_{kl} \eta_{MN} - \sum \{X_{kM}, P_{k'M'}\} \varepsilon^{k'l'} \eta^{M'N'} \left\{ \frac{1}{2} X_{l'N'}, P_{lN} \right\}, \\
&= \frac{1}{2} \varepsilon_{kl} \eta_{MN}.
\end{aligned} \tag{7.16}$$

And finally

$$\{P_{kM}, P_{lN}\}_D = \frac{1}{4} \varepsilon_{kl} \eta_{MN}. \tag{7.17}$$

Using these Dirac brackets it is easy to show $\{X_{1M}, X_{2N}\}_D = -\varepsilon_{12} \eta_{MN} = \eta_{MN}$.

We now relabel the coordinates

$$X_{1M} \equiv X_M, \quad X_{2N} \equiv P_N,$$

which gives

$$\{X_M, P_N\}_D = \eta_{MN}$$

as desired.

The instigation of Dirac brackets thus leads to a system with purely first class constraints $\phi_{ij} = 0$, ${}_A\phi_{ij} = 0$ and in which the coordinates X_M and P_N are canonically conjugate and have the usual ‘Poisson’ (actually Dirac) bracket relations. We also have the A^{ij} coordinate, and its associated canonical momentum ${}_A\pi_{ij}$.

Associated with each first class constraint is a Lagrange multiplier λ , and its conjugate momentum $\pi = 0$. We label these coordinates as

$$\begin{aligned}
\phi_{ij} &\rightarrow \lambda_{ij}, \quad {}_\lambda\pi_{ij}, \\
{}_A\phi_{ij} &\rightarrow {}_A\lambda_{ij}, \quad {}_{\lambda_A}\pi_{ij}.
\end{aligned}$$

These two new momenta are set equal to zero, and thus form new constraints, which we label ${}_\lambda\phi_{ij}$ and ${}_{\lambda_A}\phi_{ij}$ respectively. In the construction for the scalar

particle (see chapter 4) we had the constraint $\phi = \lambda\pi_\lambda$. In a similar fashion we could write the constraints

$$\begin{aligned}\lambda\phi_{ij} &= \lambda_{ik}\eta^{kl}\lambda\pi_{lj} + \lambda_{ki}\eta^{kl}\lambda\pi_{jl}, \\ \lambda_A\phi_{ij} &= {}_A\lambda_{ik}\eta^{kl}{}_{\lambda_A}\pi_{lj} + {}_A\lambda_{ki}\eta^{kl}{}_{\lambda_A}\pi_{jl}.\end{aligned}$$

However the restriction to strictly positive and constant Lagrange multipliers that made these constraints regular are no longer justified, and so we adopt the more simple identification (which is regular): $\lambda\phi_{ij} = \lambda\pi_{ij}$, $\lambda_A\phi_{ij} = \lambda_A\pi_{ij}$, which are

Finally, we introduce a conjugate pair of Grassmann odd ghosts η^{ij}, ρ_{ij} corresponding to each constraint. Each pair of ghosts obey the general equation $\{\eta^{ij}, \rho_{kl}\}_D = \delta_k^i\delta_l^j$ and have zero Dirac brackets with everything else.

In summary, the BRST-extended system consists of the following

$$\begin{array}{llllll} \text{phase space variables} & X_1^M, X_2^N & A^{ij}, {}_A\pi_{ij} & \lambda_{ij}, \lambda\pi_{ij} & {}_A\lambda_{ij}, {}_{\lambda_A}\pi_{ij} & \\ \text{constraints} & \phi_{ij} & {}_A\phi_{ij} & \lambda\phi_{ij} & {}_{\lambda_A}\phi_{ij} & \\ \text{ghosts} & \eta^{ij}, \rho_{ij} & {}_A\eta^{ij}, {}_A\rho_{ij} & \lambda\eta^{ij}, \lambda\rho_{ij} & {}_{\lambda_A}\eta^{ij}, {}_{\lambda_A}\rho_{ij} & \end{array} \quad (7.18)$$

The algebra of constraints can be written

$$\begin{aligned}\{ {}_A\phi_{ij}, {}_A\phi_{kl} \}_D &= 0, \quad \{ \lambda\phi_{ij}, \lambda\phi_{kl} \}_D = 0, \quad \{ {}_{\lambda_A}\phi_{ij}, {}_{\lambda_A}\phi_{kl} \}_D = 0, \\ \{ \phi_{ij}, \phi_{kl} \}_D &= -(\varepsilon_{jl}\phi_{ik} + \varepsilon_{ik}\phi_{lj} + \varepsilon_{il}\phi_{jk} + \varepsilon_{jk}\phi_{il}), \\ &= C_{ijkl}^{k'l'}\phi_{k'l'}. \end{aligned} \quad (7.19)$$

And the BRST operator is

$$\begin{aligned}\Omega &= \eta^{ij}\phi_{ij} + \lambda\eta^{ij}\lambda\eta^{ij}\lambda\phi_{ij} + {}_A\eta^{ij}{}_A\phi_{ij} + {}_{\lambda_A}\eta^{ij}{}_{\lambda_A}\phi_{ij} \\ &\quad - \eta^{ij}\eta^{kl}(\varepsilon_{jl}\rho_{ik} + \varepsilon_{ik}\rho_{lj} + \varepsilon_{il}\rho_{jk} + \varepsilon_{jk}\rho_{il}). \end{aligned} \quad (7.20)$$

7.2.1 Hamiltonian equations of motion and β -limiting

A system which has been quantised using the BFV-BRST formulation yields identical Hamiltonian equations of motion in the physical sector as that formulated using the Dirac method (when both methods can be applied). Similarly,

our BRST-formulation should yield the same Hamiltonian equations of motion as Bars' technique without the ghosts.

An essential ingredient of the BRST method is the choice of an appropriate gauge fixing function Ψ , which is in turn used to calculate the appropriate Hamiltonian for the system. Different forms of the gauge fixing function will result in different dynamics being exhibited by the system, and thus is equivalent to choosing the three gauge functions in the two-time formulation. Due to time limitations placed upon this thesis we have not been able to comprehensively explore the different systems that arise through this gauge fixing process, or to recover any recognisable systems through this process. Nevertheless, we shall present the work as it now stands, with the hope that we can continue with this project at a later date.

In order to recover the physical systems arrived at by Bars it is necessary to carry out the β -limiting procedure, as described in 5.5.2. This procedure allows us to set one (or more) of the Lagrange multipliers equal to some specific function and at the same time to squeeze out certain subsets of degrees of freedom, leaving over physically distinct systems. One would expect that the gauge fixing function that is used in the β -limiting procedure would suffer from Gribov problems, rendering the resultant system physically distinct from that arrived at using a different (β -limited) function. It is these Gribov ambiguities that enable different physics to arise in the two-time system.

To make the calculations easier, we shall consider the restricted case when $A^{11} = A^{12} = 0$.

$$S = \frac{1}{2} \int_0^\tau d\tau \left((\partial_\tau X_i^M) X_j^N \varepsilon^{ij} \eta_{MN} - A^{22} X_2^M X_2^N \eta_{MN} \right). \quad (7.21)$$

By solving the Euler-Lagrange equation for X_2^M , we get $X_2 = \dot{X}_1/A^{22}$. We then substitute this into the above equation, along with

$$(\partial_\tau X_2) X_1 = -(\partial_\tau X_1) X_2 + \partial_\tau (X_1 X_2) = -(\partial_\tau X_1) X_2,$$

up to boundary terms. This allows us to write the action in a form identical to

the first order action for the massless scalar particle

$$S = \frac{1}{2} \int_0^\tau d\tau \frac{(\dot{X}_1)^2}{e},$$

where we have set $A^{22} = e$ to indicate that it takes the place of the vierbein. This action is identical to the first order formulation seen in chapter 4, except in that case we had a massive particle.

In this subspace of the full physical system the Dirac brackets of the constraints ϕ_{ij} contract to $[\phi_{ij}, \phi_{kl}] = 0$, *i.e.* there are no structure constants. The BRST operator thus contracts to only the first line of equation (7.20).

We demonstrate the application of the β -limiting procedure using the gauge fixing function

$$\Psi = \frac{1}{\beta} \rho_\lambda (\lambda - \lambda_0) + \rho_\lambda + \frac{1}{\beta'} \rho_{\lambda_e} (\lambda_e - \lambda_{e0}) + \rho_e \lambda_e, \quad (7.22)$$

where λ_0 and λ_{e0} are given functions of time. The Hamiltonian can be calculated as

$$\begin{aligned} H &= \{\mathcal{F}, \Omega\} = \{\mathcal{F}, \eta(X_2 \cdot X_2) + \eta^\lambda \pi_\lambda + \eta^e \pi_e + \eta^{\lambda_e} \pi_{\lambda_e}\}, \\ &= -\frac{1}{\beta} \pi_\lambda (\lambda - \lambda_0) + \frac{1}{\beta} \rho_\lambda \eta^\lambda - \lambda(X_2 \cdot X_2) + \rho \eta^\lambda - \frac{1}{\beta'} \pi_{\lambda_e} (\lambda_e - \lambda_{e0}) \\ &\quad - \frac{1}{\beta'} \rho_{\lambda_e} \eta^{\lambda_e} - \pi_e \lambda_e + \rho_e \eta^{\lambda_e}. \end{aligned} \quad (7.23)$$

The equations of motion of the system can be determined by using $\dot{A} = \{A, H\}$. The equations which are affected by the β -limiting process are

$$\begin{aligned} \dot{\lambda} &= \frac{1}{\beta} (\lambda - \lambda_0), \\ \dot{\pi}_\lambda &= -\frac{1}{\beta} \pi_\lambda - X_2 \cdot X_2, \\ \dot{\lambda}_e &= -\frac{1}{\beta'} (\lambda_e - \lambda_{e0}), \\ \dot{\pi}_{\lambda_e} &= -\frac{1}{\beta'} \pi_{\lambda_e} - \pi_e, \\ \dot{\eta}^\lambda &= -\frac{1}{\beta} \eta^\lambda, \\ \dot{\rho}_\lambda &= -\frac{1}{\beta} \rho_\lambda - \rho, \end{aligned}$$

$$\begin{aligned}\dot{\eta}^{\lambda_e} &= -\frac{1}{\beta'}\eta^{\lambda_e}, \\ \dot{\rho}_{\lambda_e} &= -\frac{1}{\beta'}\rho_{\lambda_e} - \rho_e.\end{aligned}$$

If we now change to new variables defined through

$$\pi_\lambda = \beta \tilde{\pi}_\lambda, \quad \rho_\lambda = \beta \tilde{\rho}_\lambda, \quad \pi_{\lambda_e} = \beta' \tilde{\pi}_{\lambda_e}, \quad \rho_{\lambda_e} = \beta' \tilde{\rho}_{\lambda_e},$$

the scaled Hamiltonian can be written

$$\begin{aligned}H &= -\tilde{\pi}_\lambda(\lambda - \lambda_0) + \tilde{\rho}_\lambda \eta^\lambda - \lambda(X_2 \cdot X_2) + \rho \eta^\lambda - \tilde{\pi}_{\lambda_e}(\lambda_e - \lambda_{e0}) \\ &\quad \tilde{\rho}_{\lambda_e} \eta^{\lambda_e} - \pi_e \lambda_e + \rho_e \eta^{\lambda_e},\end{aligned}\tag{7.24}$$

whilst the above equations of motion yield the conditions

$$\begin{aligned}\lambda &= \lambda_0, & \tilde{\pi}_\lambda &= -X_2 \cdot X_2, \\ \lambda_e &= \lambda_{e0}, & \tilde{\pi}_{\lambda_e} &= \pi_e, \\ \eta^\lambda &= 0, & \tilde{\rho}_\lambda &= -\rho, \\ \eta^{\lambda_e} &= 0, & \tilde{\rho}_{\lambda_e} &= -\rho_e,\end{aligned}\tag{7.25}$$

respectively. Finally, substituting these conditions into the scaled Hamiltonian (7.24) we get

$$H = -\lambda_0 P^2 - \lambda_{e0} \pi_e,$$

where we have used $X_2 = P$. By choosing $\lambda_{e0} = 0$ and identifying the function λ_0 with the Lagrange multiplier (see chapter 4 for comments on this identification) it can be seen that this equation is indeed a form of the Hamiltonian for the massless scalar particle formulated with a vierbein. Thus in the reduced case of $A^{11} = A^{12} = 0$, with a gauge fixing function chosen to correspond to the massless scalar particle, the two-time formulation appears to give a valid result.

From this we can see that the investigation of two-time physics using a BRST approach may yield some promising results. In order to fully explore the potential of two-time physics it will be necessary to consider the general case with $A^{ij} \neq 0$. In this case a possible gauge fixing condition is

$$\mathcal{F} = \frac{1}{\beta} \lambda \rho^{ij} (\lambda_{ij} - {}_0\lambda_{ij}) + \rho^{ij} \lambda_{ij} + \frac{1}{\beta} \lambda_e \rho^{ij} ({}_e\lambda_{ij} - {}_{e0}\lambda_{ij}) + {}_e\rho^{ij} {}_e\lambda_{ij}.\tag{7.26}$$

We would expect that different gauge fixing functions would lead to different systems in one-time physics, for example the H atom, quantum oscillator *etc.*. For example, we could choose \mathcal{F} in such a way so as to enforce $A^{11} = A^{12} = 0$, and thus select the massless scalar particle system shown above. Unfortunately we have not had time to carry out more than a preliminary study of the formulation given above, however we believe it to be an interesting and potentially exciting project for future work.

In this section we have, obviously, not quantised the system. The reason for failing to do this is only a matter of convenience; the classical system with the Dirac brackets contains only first class constraints and so can be quantised by application of the rule (Poisson brackets) $\rightarrow 1/(i\hbar)$ (Commutation relations).

7.3 Quantisation of systems with second class constraints

As was stated in the previous section the BFV-BRST quantisation procedure does not directly provide for a way of quantising systems with second class constraints. Many physical systems contain second class constraints (two-time physics being just one example) and so a generalisation of the BRST method to second class constraints is an important step forward. The introduction of Dirac brackets, as was carried out in the previous section, re-expresses the system as one without second class constraints. Another technique for removing second class constraints from general systems was proposed by Berezin [81]. Both these methods were proposed before BFV-BRST quantisation was discovered and so do not take into account any of the inherent properties of the BRST method.

In the years since BFV-BRST quantisation was put forward several attempts have been made to generalise the method to systems involving second class constraints. The majority of these attempts fall into two distinct categories.

- The first approach is to enlarge the phase space still further in such a way

that the second class constraints become effectively first class constraints. After this step is taken the usual BRST method is applied. The most common way of doing this is via the method proposed by Batalin, Fradkin and Fradkina (BFF) [82, 83, 84]. An alternative method is presented in [85].

- The second approach does not introduce new variables beyond those that already exist in the BRST extended phase space, but instead assumes certain structures on the second class constraints, which are used to change the ghost structure and in this way remove the second class constraints [86, 87, 88, 89, 90, 91, 92].

A discussion of the application of the second of these two methods in a general system is given in [93]. This reference uses the 4D superparticle as an example. An example of the BFF method is given in [94]. This paper also contains a long list of references of alternative applications of the BFF method.

In the remainder of this section we shall sketch the beginning steps of the BFF method by following the procedure given in [84] for the two-time particle defined in the previous section.

Given the classical (non BRST) system constructed in the previous sections, we can quantise the system as follows. We start with the canonical conjugate pairs of dynamical variables in the original phase space

$$(X_i, P^i) \quad (A^{ij}, {}_A\pi_{ij}), \quad i, j = 1, 2.$$

Along with this we have a Hamiltonian $H_0(X, P, A, {}_A\pi)$ defined through the action. We also have the first class constraints $\phi_{ij}, {}_A\phi_{ij}$, which for convenience we shall write as

$$\Phi_a = (\phi_{ij}, {}_A\phi_{ij}),$$

where $a = 1, \dots, 6$ (remembering that the constraints are symmetric). The second class constraints were defined in (7.12), we rewrite them here for clarity

$$\chi_i(X, P, A, {}_A\pi) = P_i + \frac{1}{2}\varepsilon_{ij}X^j.$$

Let us now define the new Grassmann even, operators Ξ^i such that

$$[\Xi_i, \Xi_j] = i\hbar\omega_{ij},$$

where ω_{ij} is a constant symplectic matrix, $\omega_{ij} = -\omega_{ji}$. We further define the Grassmann even operators $T_i(X, P, A, {}_A\pi, \Xi)$ by the equation

$$[T_i, T_j] = 0, \quad (7.27)$$

and the condition

$$T_i(X, P, A, {}_A\pi, 0) = \chi_i(X, P, A, {}_A\pi). \quad (7.28)$$

Equations (7.27) and (7.28) convert the second class constraint χ_i in the original phase space into an effective abelian first class constraint T_i in the phase space $(X_i, P^i, A^{ij}, {}_A\pi_{ij}, \Xi_i)$

We have now constructed an extended phase space with only first class constraints. This system may then be quantised using the standard BFV-BRST prescription given in chapter 3, *i.e.* introducing ghosts (η, ρ) , defining the BRST operator $\Omega(X, P, A, {}_A\pi, \Xi, \eta, \rho)$ and gauge fixing condition leading to a BRST extended Hamiltonian $H_B(X, P, A, {}_A\pi, \Xi, \eta, \rho)$. The BFF method uses the following equations to define the BRST operator and Hamiltonian

$$\begin{aligned} \{\Omega, \Omega\} &= 0, & [\Omega, T_i] &= 0, \\ [H_B, \Omega] &= 0, & [H_B, T_i] &= 0, \end{aligned}$$

and the conditions

$$\Omega(X, P, A, {}_A\pi, 0, \eta, 0) = \Phi_a(X, P, A, {}_A\pi)\eta^a, \quad (7.29)$$

$$H_B(X, P, A, {}_A\pi, 0, 0, 0) = H_0(X, P, A, {}_A\pi). \quad (7.30)$$

The remaining crucial step in the BFF method is to determine an explicit value for the first class constraint T_i . In their paper [84] BFF state how this may be done, however in the case of the two-time particle it appears sufficient to define T_i as

$$T_i = \chi_i + \Xi_i = P_i + \frac{1}{2}\varepsilon_{ij}X^j + \phi_i,$$

where we must also identify $w_{ij} = \varepsilon_{ij}$. If this is the case then we automatically satisfy (7.28) as well as (7.27)

$$\begin{aligned}
[T_i, T_j] &= [P_i + \frac{1}{2}\varepsilon_{ik}X^k + \Xi_i, P_j + \frac{1}{2}\varepsilon_{jl}X^l + \Xi_j], \\
&= \frac{1}{2}\varepsilon_{ik}[X^k, P_j] + \frac{1}{2}\varepsilon_{jl}X^l[P_i, X^l] + [\Xi_i, \Xi_j], \\
&= i\hbar(-\frac{1}{2}\varepsilon_{ij} + \frac{1}{2}\varepsilon_{ji} + \varepsilon_{ij}), \\
&= 0.
\end{aligned}$$

Due to time restrictions, we have gone no further than this in our BRST investigation of the two-time system. The work done by Bars [70, 71, 72, 73, 74, 75, 76, 77, 4, 78, 79] has been far reaching, yet he appears to have taking liberties in some areas. For example he conveniently sidesteps the issue of second class constraints in his formulation by sleight of hand and a convenient use of boundary terms. This leads to his claim that $X_1^M(\tau)$ and $X_2^M(\tau)$ are a conjugate pair, which we have shown to be untrue. Secondly, Bars has presented several examples of diverse one-time systems that can be arrived at through his process but has not given any indication of how the appropriate gauge fixing choices were arrived at. If the theory is to be successful there must be some method of determining a gauge fixing condition that will lead to a given one-time system.

We believe that a thorough exploration of the two-time system from a different perspective is necessary to justify Bars' work. Furthermore, we believe that the BFV-BRST formulation is a good approach, and that the work we have done in this area is rigorous and a step in the right direction. That being said much work remains to be done.

The major obstacle in any formulation of Bars' two-time physics is the presence of the second class constraints (7.12). In section 7.2 we used Dirac brackets to re-express the system such that it only contained first class constraints. This line seems promising; the fundamental problem with Dirac brackets is trying to find an explicit representation for them. In the case of two-time physics this representation is easily found, and so we assume that the remainder of the process will go smoothly. Finding an appropriate gauge fixing condition which leads to a

recognisable Hamiltonian may yet prove to be a problem. As can be seen in section 7.2, the choice of gauge fixing condition which leads to the scalar particle is not obvious, and even once we have chosen a particular Ψ we will most probably need to employ the β -limiting process in order to arrive at a physical solution. All of these considerations lead us to entertain alternative methods, such as those outlined in this section.

The advantage of the BFF method is that it has been tailored specifically to BFV-BRST quantise a system with second class constraints. Furthermore, there have been numerous successful implementations of the procedure, as can be seen in the references of [94]. Combined with this history of success, the method also gives rules for choosing a general Hamiltonian, which may guide us in our specific choice of gauge fixing conditions necessary to resolve particular models. As a result of these two considerations we believe that the BFF method will, in time, yield a fully quantised and consistent two-time physics system. Once we have this system we shall be able to begin to verify the claims made by Bars about the properties of two-time physics.

Chapter 8

Conclusion

The aim of this thesis was to increase our understanding of gauge symmetries, along with their graded extensions and the theory of their associated representations in the hope that this may enable admissible quantisation schemes to be implemented covariantly (and systematically) at an algebraic level. In order to achieve this aim we have adopted the attitude that the general principle of this algebraic version of the quantisation program should emerge from detailed consideration of particular case studies. The majority of original work presented in this thesis consists of the presentation of three such specific examples; the scalar particle (chapter 4), the spinning particle (chapter 5) and the $D(2, 1; \alpha)$ particle (chapter 6), all of which were propagating in flat Minkowski space-time. We have also applied many of the same techniques to a study of two-time physics (chapter 7).

In the case of the scalar and spinning particles, the enlarged algebra needed to accommodate the global supersymmetries under which gauge and ghost degrees of freedom transform, as well as the symmetries of the constraint algebra and other symmetries possessed by the system, was shown to be the orthosymplectic extension of the Poincaré space-time supersymmetry algebra $iosp(d, 2/2)$.

In chapter 4 we used an appropriate definition of the generators J_{MN} , P_N , and the introduction of the BRST operator, to show that our superalgebraisation of the BFV-BRST quantisation of the scalar particle gave the correct scalar rep-

representation of $iosp(d, 2/2)$ as carried on the space of covariant solutions of the massive Klein-Gordon equation. We could then show that this representation yielded an identical state space structure to that of the standard Hamiltonian BFV-BRST *Ansatz* for the scalar particle.

Due to the presence of anti-commuting coordinates in the action of the spinning particle, the generators J_{MN} had both orbital and spin components. The spin components corresponded to an extended Clifford algebra with generators entailing both Bosonic and Fermionic oscillators. This allowed us to effect a decomposition of the representation space and split the Dirac wavefunctions so that the $iosp(d, 2/2)$ algebra was effectively realised on $2^{d/2}$ -dimensional Dirac spinors. In order to show that the obtained representation had an identical state space structure to that of the standard Hamiltonian BFV-BRST *Ansatz* for the spinning particle, the physical model's extended phase space had to be contracted using the β -limiting process.

In chapter 6 a different approach was taken from the previous two chapters. The exceptional superalgebra $D(2, 1; \alpha)$ is a generalisation of the classical simple Lie superalgebra $osp(d, 2/2)$, for $d = 2$. Through an appropriate generalisation we have shown that it is possible to convert the quantisation superalgebra from $osp(2, 2/2)$ to $D(2, 1; \alpha)$. This was achieved without a corresponding classical physical model of a particle, and so relied entirely on the algebraic structure. Once the appropriate quantisation superalgebra had been achieved we reverse-engineered, by way of a Legendre transformation, the corresponding classical action. This action corresponded to a massless scalar particle in a potential containing two parts; a simple oscillator term, and a term containing a cross product between position and velocity.

The scalar (spin $s = 0$) and spinning ($s = \frac{1}{2}$) particles are among the simplest of supersymmetric particles that one can consider. However the techniques we have developed in this thesis, and the success that we have had, has laid the groundwork for significant advances along these lines in the near future. An obvious extension of this work is to carry out our program on a particle of arbitrary

higher spin s . This would entail, at the level of relativistic dynamics, starting with $2s$ additional Grassman coordinates $\zeta_\mu^{(i)}, \zeta_5^{(i)}$ with Bargmann-Wigner [95, 96] type first class constraints

$$\phi^{(i)} = \zeta^{\mu(i)} p_\mu - m \zeta_5^{(i)}.$$

At the level of representations, the starting point is a reducible product representation of the $osp(d, 2/2)$ spinor valued superfield of the spinning case.

In a slightly different direction, the spinning particle is identical to supergravity in one dimension (with x^μ being replaced by the field ϕ^μ), and so having established this result for $1D$ supergravity, an extension to higher dimensions should be within our grasp. We are also considering how the method can be applied to superstring or superparticle cases, for which a covariant approach has so far been problematical [97].

Chapter 6 demonstrated that for $d = 2$ it is possible to formulate the quantisation superalgebra in terms of generalisations of $iosp(d, 2/2)$. A logical extension of this work is to find other generalisations of $iosp(d, 2/2)$ (or the homogeneous $osp(d, 2/2)$). For example in $d = 2 + 1$ we have $osp(3, 2/2)$ which corresponds to anti de Sitter symmetry (which may thus be relevant to anyon quantisation), and for $d = 3 + 1$ we get conformal symmetry of $4D$ spacetime and super unitary superalgebras as possible alternative quantisation superalgebras. Then there are the two remaining exceptional Lie superalgebras; $F(4)$, which applies when $d = 4 + 1$, and $G(3)$. $F(4)$ has dimension 40, its even part is a non-compact form of $sl(2) \oplus o(7)$ (24-dimensional), whilst the odd part (of dimension 16) is the spinorial representation of $sl(2) \oplus o(7)$. The 31-dimensional $G(3)$ has its even part as a 17-dimensional non-compact form of $sl(2) \oplus G_2$ whilst the odd part is the 14-dimensional spinorial representation of $sl(2) \oplus G_2$.

Conformal supersymmetry has long been of interest as a probable higher symmetry underlying particle interactions, no more so than in the light of recent interpretations of compactifications of higher dimensional supergravities [98, 99]. The spinning particle is of particular interest in this respect [100] as the traditional descent from $d + 2$ to d dimensions - via a projective conformal space [53]

- is here implemented not on the cone (massless irreducible representations), but for the massive super-hyperboloid. The method used in chapter 4 and 5 can also be seen as an elaboration of the method of conformalisation [28]. It has been shown [101] that the superalgebra that unifies BRST and conformal symmetry is $osp(d+1, 3/2)$, and so a covariant representation of $osp(d+1, 3/2)$ utilising this unification should be possible [102]. Beyond the Dirac equation and higher spin generalisations, this thesis has also laid the foundations of investigations into the algebraic BFV-BRST complex associated with indecomposable representations [103, 104] (for example, where the vector-scalar super-special conformal generators are represented as nilpotent matrices).

Finally, in chapter 7, we examined the formalism of two-time physics, as proposed by Bars [4], from a BRST perspective. Much of Bars' work has dealt with the formulation of physical theories in two-time form. He has not, however, closely examined the problem of quantisation of a classical two-time system. Chapter 7 thus sought to apply the standard BFV-BRST quantisation techniques to such a system. This process ran into trouble when we discovered that the system contained second class constraints, a point not emphasized by Bars. In the chapter we made inroads into removing the second class constraints (as is necessary so as to apply the BFV-BRST method) but have not yet been successful. Thus this chapter remains unfinished, however we believe that with further work we will be able to attain our goal. In summary, Bars has made some far reaching claims, yet he has taken liberties in his notation, and not explained several key aspects. We believe that an investigation, such as that we have begun, is necessary to verify the claims made by Bars about two-time physics.

Appendix A

(a, b) representation of physical states

A.1 Preliminary construction

We can define a Heisenberg-like algebra as follows

$$\begin{aligned} [a, b^\dagger] &= 1 = [b, a^\dagger], \\ [a, b] &= 0 = [b, a], \\ [a, a^\dagger] &= 0 = [b, b^\dagger], \end{aligned} \tag{A.1}$$

where we can take $a|0, 0\rangle = 0 = b|0, 0\rangle$, and define

$$|m, n\rangle = (a^\dagger)^m (b^\dagger)^n |0, 0\rangle, \quad m, n \geq 0, \tag{A.2}$$

note that this implies

$$\begin{aligned} \langle 0, 1 | 1, 0 \rangle &= \langle 0, 0 | b a^\dagger | 0, 0 \rangle = 1, \\ \langle 1, 0 | 0, 1 \rangle &= 1. \end{aligned} \tag{A.3}$$

In fact, in general we have

$$\langle m', n' | m, n \rangle = m! n! \delta_{m'n} \delta_{n'm}, \tag{A.4}$$

and so we redefine our basis by

$$\begin{aligned} |m, n\rangle' &= a^{\dagger m} b^{\dagger n} |0, 0\rangle, \\ |m, n\rangle &= \frac{1}{\sqrt{m! n!}} |m, n\rangle' = \frac{(a^\dagger)^m (b^\dagger)^n}{\sqrt{m! n!}} |0, 0\rangle. \end{aligned} \tag{A.5}$$

A.2 Realisation of $\zeta_\alpha, \hat{\zeta}_\alpha$

As explained in section 5.3, the operators $\zeta_\alpha, \tilde{\zeta}_\alpha$ are constructed using a two-dimensional Bosonic oscillator algebra (a, b) . We choose to define $\zeta_\alpha, \tilde{\zeta}_\alpha$ as follows

$$\begin{aligned}\zeta_\alpha &= \frac{1}{\sqrt{2}} ((b \pm a) - (b^\dagger \mp a^\dagger)), \\ \tilde{\zeta}_\alpha &= \frac{1}{\sqrt{2}} ((a \pm b) - (a^\dagger \mp b^\dagger)).\end{aligned}\tag{A.6}$$

At the same time we can define the ghost state η^α of section 5.5, and its conjugate momentum ρ_α as

$$\begin{aligned}\eta^1 &= \frac{i}{\sqrt{2}}(a - a^\dagger), \quad \eta^2 = \frac{i}{\sqrt{2}}(b - b^\dagger), \\ \rho_1 &= \frac{1}{\sqrt{2}}(b + b^\dagger), \quad \rho_2 = \frac{1}{\sqrt{2}}(a + a^\dagger).\end{aligned}\tag{A.7}$$

Equation (A.6), together with the spinors used in section 5.4 allow us to write

$$\begin{aligned}(\eta \cdot \zeta) &= i(b - b^\dagger) = \sqrt{2}\eta_2, \\ (\eta' \cdot \zeta) &= -(a + a^\dagger) = -\sqrt{2}\rho_2.\end{aligned}\tag{A.8}$$

In Section 5.4 the eigenstates of $(\eta \cdot \zeta)(\eta' \cdot \zeta)$, with some eigenvalue Λ were required in the analysis of the physical states. From equation (A.6) we can write

$$(\eta \cdot \zeta)(\eta' \cdot \zeta) = i(a^\dagger b^\dagger - ab + ab^\dagger - a^\dagger b).\tag{A.9}$$

We have

$$a^\dagger b^\dagger |m, n\rangle = \sqrt{(m+1)(n+1)} |m+1, n+1\rangle,\tag{A.10}$$

and

$$\begin{aligned}ab |m, n\rangle &= \frac{b^{\dagger m} a b^{\dagger n}}{\sqrt{m!n!}} |m+1, n+1\rangle, \\ &= \frac{[b, a^{\dagger m}][a, b^{\dagger n}]}{\sqrt{m!n!}} |m, n\rangle, \\ &= \sqrt{mn} |m-1, n-1\rangle.\end{aligned}\tag{A.11}$$

Similarly

$$a^\dagger b |m, n\rangle = \frac{a^\dagger [b, a^{\dagger m}] a^{\dagger n}}{\sqrt{m!n!}} |m, n\rangle = m |m, n\rangle,\tag{A.12}$$

$$ab^\dagger |m, n\rangle = \frac{a^{\dagger m} [a, b^{\dagger(n+1)}]}{\sqrt{m!n!}} |m, n\rangle = (n+1) |m, n\rangle.\tag{A.13}$$

As $(\eta \cdot \zeta)(\eta' \cdot \zeta)$ commutes with $(a^\dagger b - b^\dagger a)$ (the false ghost number), we specialise to eigenstates $|\Lambda\rangle$ with $m = n$

$$|\Lambda\rangle = \sum_0^\infty \Lambda_m |m, m\rangle, \quad (\text{A.14})$$

therefore

$$\begin{aligned} (\eta \cdot \zeta)(\eta' \cdot \zeta)|\Lambda\rangle &= i(a^\dagger b^\dagger - ab + ab^\dagger - a^\dagger b)|\Lambda\rangle, \\ &= i \sum_{m=0}^\infty [(m+1)\Lambda_m |m+1, m+1\rangle - \Lambda_m |m, m\rangle \\ &\quad - m\Lambda_m |m-1, m-1\rangle], \\ &= i[\Lambda_0 |1, 1\rangle - \Lambda_0 |0, 0\rangle + 2\Lambda_1 |2, 2\rangle - \Lambda_1 |1, 1\rangle - \Lambda_1 |0, 0\rangle \\ &\quad + 3\Lambda_2 |3, 3\rangle - \Lambda_2 |2, 2\rangle - 2\Lambda_2 |1, 1\rangle + \dots], \\ &= i[-(\Lambda_0 + \Lambda_1)|0, 0\rangle + (\Lambda_0 - \Lambda_1 - 2\Lambda_2)|1, 1\rangle \\ &\quad + (2\Lambda_1 - \Lambda_2 - 3\Lambda_3)|2, 2\rangle \\ &\quad + \dots + (m\Lambda_{m-1} - \Lambda_m - (m+1)\Lambda_{m+1})|m, m\rangle + \dots], \\ &= \Lambda(\Lambda_0 |0, 0\rangle + \Lambda_1 |1, 1\rangle + \dots + \Lambda_m |m, m\rangle + \dots \end{aligned}$$

And so

$$\begin{aligned} -i(\Lambda_0 + \Lambda_1) &= \Lambda\Lambda_0, \\ i(\Lambda_0 - \Lambda_1 - 2\Lambda_2) &= \Lambda\Lambda_1, \\ i(2\Lambda_1 - \Lambda_2 - 3\Lambda_3) &= \Lambda\Lambda_2, \end{aligned}$$

or in general

$$i(m\Lambda_{m-1} - \Lambda_m - (m+1)\Lambda_{m+1}) = \Lambda\Lambda_m. \quad (\text{A.15})$$

Re-expressing these in terms of Λ and Λ_0 only we get

$$\begin{aligned} \Lambda_1 &= (i\Lambda - 1)\Lambda_0, \\ \Lambda_2 &= -\frac{1}{2}(\Lambda^2 + 2i\Lambda - 2)\Lambda_0, \\ \Lambda_3 &= \frac{1}{6}(-i\Lambda^3 + 3\Lambda^2 + 8i\Lambda - 6)\Lambda_0, \\ \Lambda_4 &= \frac{1}{24}(\Lambda^4 + 4i\Lambda^3 - 20\Lambda^2 - 32i\Lambda + 24)\Lambda_0, \end{aligned} \quad (\text{A.16})$$

$$\Lambda_5 = \frac{i}{120}(\Lambda^5 + 5i\Lambda^4 - 40\Lambda^3 - 100i\Lambda^2 + 184\Lambda + 120i)\Lambda_0.$$

etc.

It is easy to write a short program to generate Λ_m to any order.

The algebra of the Bosonic ghost oscillator modes is a non-conventional Heisenberg algebra [105] with an indefinite inner product. The algebra provides a realisation of $O(3, 2)$ through a bilinear form, the generators of which are as follows: first, we define the operators K^0, K^+ and K^- as

$$K^0 = \frac{1}{2}(a^\dagger b + ab^\dagger), \quad K^+ = a^\dagger b^\dagger, \quad K^- = ab,$$

and

$$\begin{aligned} A^+ &= (a^\dagger)^2, & A^0 &= a^\dagger a, & A^- &= a^2, \\ B^+ &= (b^\dagger)^2, & B^0 &= b^\dagger b, & B^- &= b^2. \end{aligned}$$

The 10 generators of $O(3, 2)$ can then be calculated to be

$$\begin{aligned} J_{12} &= iK^0, & J_{31} &= \frac{1}{2}(K^+ - K^-), & J_{23} &= \frac{i}{2}(K^+ + K^-), \\ J_{14} &= \frac{i}{2}(A^+ + A^-), & J_{24} &= \frac{1}{2}(A^+ - A^-), & J_{34} &= iA^0, \\ J_{15} &= \frac{i}{2}(B^+ + B^-), & J_{25} &= \frac{1}{2}(B^+ - B^-), & J_{35} &= iB^0, \\ J_{45} &= \frac{1}{2}(a^\dagger b - ab^\dagger). \end{aligned} \tag{A.17}$$

We define two diagonal operators; the ghost number operator $N_{\text{gh}} = i(a^\dagger b^\dagger - ab + ab^\dagger - a^\dagger b)$ and the false ghost number operator $\overline{N}_{\text{gh}} = a^\dagger b - b^\dagger a$. In the number basis $|m, n\rangle$ the false ghost number operator has eigenvalue $(m - n)$ whilst the ghost number operator has, of course, the eigenvalue Λ , for any complex $|\Lambda\rangle$.

In general, this eigenstates in this space are not normalizable, however Wünsche [106] has studied the spectrum of linear combinations of the oscillator modes $a + \varpi a^\dagger$, for arbitrary complex number ϖ , and identified formal eigenvalues within a Gel'fand triple structure $K \in H \in K'$, where K' is the rigged Hilbert space. Due to time constraints we have not pursued this work any further, thus we cannot provide any firm conclusions.

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